Algebr Represent Theor (2016) 19:783–826
DOI 10.1007/s10468-016-9599-2

Broué's Perfect Isometry Conjecture Holds for the Double Covers of the Symmetric and Alternating Groups

Michael Livesey[1]

Received: 19 August 2014 / Accepted: 26 January 2016 / Published online: 3 March 2016

Abstract In O. Brunat and J. Gramain (2014) recently proved that any two blocks of double covers of symmetric groups are Broué perfectly isometric provided they have the same weight and sign. They also proved a corresponding statement for double covers of alternating groups and Broué perfect isometries between double covers of symmetric and alternating groups when the blocks have opposite signs. Using both the results and methods of O. Brunat and J. Gramain in this paper we prove that when the weight of a block of a double cover of a symmetric or alternating group is less than p then the block is Broué perfectly isometric to its Brauer correspondent. This means that Broué's perfect isometry conjecture holds for the double covers of the symmetric and alternating groups.We also explicitly construct the characters of these Brauer correspondents which may be of independent interest to the reader.

Keywords Representation theory · Broue's perfect isometry conjecture · Double covers or symmetric and alternating groups · Finite group theory

Mathematics Subject Classification (2010) 20C15

1 Introduction

Broué's abelian defect group conjecture postulates that every block with abelian defect is derived equivalent to its Brauer correspondent. The conjecture was proved for symmetric groups in a combination of two papers by J. Chuang and R. Kessar [4, Theorem 2] and J.

Presented by Radha Kessar.

✉ Michael Livesey
michael.livesey@manchester.ac.uk

1 School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK

Chuang and R. Rouquier [5, Theorem 7.2]. Let p be a prime and recall that to each p-block of a symmetric group is associated a weight w. The defect group is abelian if and only if $w < p$. The first of these two papers proved that for any weight $w < p$ there exists a block of a symmetric group of weight w that is Morita equivalent to its Brauer correspondent. The second paper proved that any two blocks of, possible different, symmetric groups with the same weight are derived equivalent.

Presently an analogue of [4, Theorem 2] does not exist for the double covers of symmetric groups. However, recently O. Brunat and J. Gramain proved an analogue of [5, Theorem 7.2] at the level of characters for the double covers (see [2, Theroem 4.21]). In other words they proved that any two blocks of double covers of symmetric groups with the same weight and sign are Broué perfectly isometric. They also proved a corresponding statement for double covers of alternating groups and Broué perfect isometries between double covers of symmetric and alternating groups when the blocks have opposite signs. The main notion they employed in their proof was that of an MN-structure. This allows one to express character values of a block of a double cover of a symmetric group in terms of character values of blocks of smaller double covers (originally a result of M. Cabanes that was derived from Morris' formula, see [11, 12] and [3, Thoerem 20]). The MN-structures of two blocks of the same weight and sign commute with an isometry between these two blocks and this then allows one to prove that this isometry is in fact a Broué perfect isometry between the two blocks.

In this paper we develop an MN-structure for the Brauer correspondent of a block of a double cover. This involves very explicitly constructing the characters of such a group. This is a new result (see Theorem 7.12). Using this MN-structure we then go on to prove our main Theorem.

Theorem 1.1 *Let p be an odd prime, n a positive integer and B a p-block of $\tilde{S}_n$ or $\tilde{A}_n$ with abelian defect group. Then there exists a Broué perfect isometry between B and its Brauer correspondent.*

In other words, Broué's perfect isometry conjecture holds for the double covers of the symmetric and alternating groups.

We now give a brief description of each section. Section 2 contains all the results on MN-structures and generalised perfect isometries we need from [2] as well as a couple of lemmas on Clifford theory. Section 3 describes all the relevant combinatorics needed before we introduce the symmetric groups and their double covers in Section 4. Clifford algebras are needed to construct the characters of parabolic subgroups of the double covers. We introduce Clifford algebras in Section 5 and parabolic subgroups and their characters in Section 6. Section 7 introduces the group $\tilde{N}_p^t\tilde{S}_t$ and very explicitly constructs its characters. We then go on to prove some more detailed lemmas about the character values of $\tilde{N}_p^t\tilde{S}_t$ in Section 8. The MN-rules used in the MN-structures of $\tilde{S}_n$ and $\tilde{N}_p^t\tilde{S}_t$ are described in Section 9 before we prove Theorem 1.1 in Section 10.

2 Preliminaries

Before we go on to look at the specific groups that are are the subject of this paper we need some general preliminaries on representation theory.

2.1 MN-Structures and Generalised Perfect Isometries

In this section we define the notion of MN-structure as introduced by Brunat and Gramain in [2]. (For more details see [2, §2].) This will be the main tool used in proving Theorem 1.1. Let G be a finite group. We have the usual inner product

$$\langle \chi, \psi \rangle_G = \frac{1}{|G|} \sum_g \chi(g)\overline{\psi(g)}$$

for $\chi, \psi \in \mathbb{C}\,\mathrm{Irr}(G)$. Let C be a union of conjugacy classes of G. We define the map

$$\begin{aligned} \mathrm{res}_C : \mathrm{Irr}(G) &\to \mathrm{Irr}(G) \\ \mathrm{res}_C(\chi)(g) &= \begin{cases} \chi(g) & \text{if } g \in C, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consider the equivalence relation on $\mathrm{Irr}(G)$ generated by $\chi \sim \psi$ if $\langle \mathrm{res}_C(\chi), \mathrm{res}_C(\psi) \rangle_G \neq 0$. We define the C-blocks of G to be the equivalence classes of this relation. If B is a union of C-blocks then we set $\mathrm{Irr}(B)^C = \{\mathrm{res}_C(\chi) | \chi \in \mathrm{Irr}(B)\}$ and if b is a $\mathbb{C}$-basis of $\mathbb{C}\,\mathrm{Irr}(B)^C$ then we denote by $b^\vee$ the dual basis of $\mathbb{C}\,\mathrm{Irr}(B)^C$ with respect to $\langle , \rangle_G$. In other words $b^\vee = \{\Phi_\psi\}_{\psi \in b}$ where $\langle \psi, \Phi_\theta \rangle_G = \delta_{\psi,\theta}$. If, in addition, b is a $\mathbb{Z}$-basis of $\mathbb{Z}\,\mathrm{Irr}(B)^C$ then we define the integers $d_{\chi,\psi}$ via the equation

$$\mathrm{res}_C(\chi) = \sum_{\psi \in b} d_{\chi,\psi} \psi \text{ for all } \chi \in Irr(B).$$

Lemma 2.1 [2, Corollary 2.3] *With the above notation*

$$\Phi_\psi = \sum_{\psi \in b} d_{\chi,\psi} \chi \textit{ for all } \chi \in b.$$

Moreover,

$$\mathbb{Z}\,\mathrm{Irr}(B) \cap \mathbb{Z}\,\mathrm{Irr}(G)^C = \mathbb{Z}b^\vee.$$

We now define the notion of an MN-structure.

Definition 2.2 [2, Definition 2.5] Let G be a finite group, C a union of conjugacy classes with $1 \in C$ and B a union of C-blocks. We say G has an MN-structure with respect to C and B if the following hold:

1. There exists a union of conjugacy classes S of G with $1 \in S$.
2. There exists a subset $A \subset C \times S$ and a bijection $A \to G$ given by $(x_C, x_S) \mapsto x_C x_S = x_S x_C$. Furthermore, $(x_C, 1), (1, x_S) \in A$ for all $x_C \in C$ and $x_S \in S$ and also $({}^g x_C, {}^g x_S) \in A$ for all $(x_C, x_S) \in A$ and $g \in G$.
3. For all $x_S \in S$ there exists $G_{x_S} \leq C_G(x_S)$ such that $C \cap G_{x_S} = \{x_C \in C | (x_C, x_S) \in A\}$.
4. For all $x_S \in S$ there exists a union of $(G_{x_S} \cap C)$-blocks B_{x_S} and a linear map $r^{x_S} : \mathbb{C}\,\mathrm{Irr}(B) \to \mathbb{C}\,\mathrm{Irr}(B_{x_S})$ such that

$$r^{x_S}(\chi)(x_C) = \chi(x_C x_S) \quad \text{for all} \quad \chi \in B \text{ and } (x_C, x_S) \in A.$$

Also $G_1 = G$, $B_1 = B$ and $r^1 = \mathrm{id}$.

For the rest of this section we assume G is a finite group with an MN-structure and adopt the notation of Definition 2.2. For each $x_S \in S$ we define

$$\begin{aligned} d_{x_S} \quad &: \mathbb{C}\,\mathrm{Irr}(B) \to \mathbb{C}\,\mathrm{Irr}(B_{x_S}) \\ d_{x_S}\ (\chi) &= \mathrm{res}_C \circ r^{x_S}(\chi) \text{ for all } \chi \in \mathrm{Irr}(B) \end{aligned}$$

and extend linearly. We also let $e_{x_S} : \mathbb{C}\,\mathrm{Irr}(B_{x_S}) \to \mathrm{Irr}(B)$ be the adjoint of d_{x_S}. In other words

$$\langle e_{x_S}(\chi), \psi\rangle_G = \langle \chi, d_{x_S}(\psi)\rangle_{G_{x_S}} \text{ for all } \chi \in \mathbb{C}\,\mathrm{Irr}(B_{x_S}) \text{ and } \psi \in \mathbb{C}\,\mathrm{Irr}(B).$$

Take any $y = y_S y_C \in G$ with $y_S \in x_S^G$. For any $t \in G$ such that ${}^t y_S = x_S$, one has ${}^t y_C \in G_{x_S}$ by Definition 2.2, and the set X of elements ${}^t y_C$ with $t \in G$ such that ${}^t y_S = x_S$ is stable under G_{x_S}-conjugation (because $G_{x_S} \leq C_G(x_S)$). We denote by $\mathcal{E}^{x_S}_{y_S, y_C}$ a set of representatives of the G_{x_S}-classes of X.

Now let $S = \cup_{\lambda\in\Lambda}\lambda$ where each λ is a conjugacy class of G and pick a set $\{s_\lambda\}_{\lambda\in\Lambda}$ of representatives for the conjugacy classes in Λ. For each $\lambda \in \Lambda$ we define $G_\lambda := G_{s_\lambda}$, $B_\lambda := B_{s_\lambda}$, $r^\lambda := r^{s_\lambda}$, $d_\lambda := d_{s_\lambda}$ and $e_\lambda := e_{s_\lambda}$. Futhermore, if $g = g_S g_C \in G$ and $g_S \in \lambda$, we set $\mathcal{E}^{\lambda}_{g_S, g_C} := \mathcal{E}^{x_\lambda}_{g_S, g_C}$ and define

$$\begin{aligned} l_\lambda : \mathbb{C}\,\mathrm{Irr}(G_\lambda)^C &\to \mathbb{C}\,\mathrm{Irr}(G) \\ l_\lambda(\chi)(g) &= \begin{cases} \frac{1}{|\mathcal{E}^\lambda_{g_S, g_C}|}\sum_{x_C \in \mathcal{E}^\lambda_{g_S, g_C}} \chi(x_C) & \text{if } g_S^G = \lambda \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We also pick a $\mathbb{Z}$-basis b_λ for $\mathbb{Z}\,\mathrm{Irr}(B_\lambda)^{C\cap G_\lambda}$ and set $b_\lambda^\vee = \{\Phi_\psi | \psi \in b_\lambda\}$ to be the dual basis of b_λ. Let G' be a finite group, C' a union of conjugacy classes and B' a union of C'-blocks of G'. Consider the isomorphism

$$\begin{aligned} \theta : \mathbb{C}\mathrm{Irr}(B) \otimes \mathbb{C}\mathrm{Irr}(B') &\to \mathrm{End}(\mathbb{C}\mathrm{Irr}(B), \mathbb{C}\mathrm{Irr}(B')) \\ \chi \otimes \chi' \quad &\mapsto (\psi \mapsto \langle \psi, \overline{\chi}\rangle \chi'). \end{aligned}$$

We note that if $f \in \mathrm{End}(\mathbb{C}\,\mathrm{Irr}(B), \mathbb{C}\,\mathrm{Irr}(B'))$ then

$$\hat{f} := \Theta^{-1}(f) = \sum_{j=1}^{r} \overline{e_j^\vee} \otimes f(e_j)$$

for some basis $(e_1, \ldots, e_r)$ of $\mathbb{C}\,\mathrm{Irr}(B)$.

Theorem 2.3 [2, Theorem 2.10] *Let G and G' be two finite groups. Suppose that*

1. *The group G (respectively G') has an MN-structure with respect to C and B (respectively C' and B'). We keep the same notation as above, and the objects relative to G' are denoted with a "prime".*
2. *Assume there are subsets $\Lambda_0 \subset \Lambda$ and $\Lambda_0' \subset \Lambda'$ such that:*
   (a) *For every $\lambda \in \Lambda\backslash\Lambda_0$ (respectively $\lambda' \in \Lambda'\backslash\Lambda_0'$), we have $r^\lambda = r'^{\lambda'} = 0$.*
   (b) *There is a bijection $\sigma : \Lambda_0 \to \Lambda_0'$ with $\sigma(\{1\}) = \{1\}$ and for $\lambda \in \Lambda_0$, an isometry $I_\lambda : \mathbb{C}\,\mathrm{Irr}(B_\lambda) \to \mathbb{C}\,\mathrm{Irr}(B'_{\sigma(\lambda)})$ such that $I_\lambda \circ r^\lambda = r'^{\sigma(\lambda)} \circ I_{\{1\}}$.*
3. *For $\lambda \in \Lambda_0$, we have $I_\lambda(\mathbb{C}b_\lambda^\vee) = \mathbb{C}b'^\vee_{\sigma(\lambda)}$.*

We write $J_\lambda = I_\lambda|_{\mathbb{C}b_\lambda^\vee}$ *and* J_λ^* *for the adjoint of* J_λ. *Then for all* $x \in G$, $x' \in G'$, *we have*

$$\hat{I}_{\{1\}}(x, x') = \sum_{\lambda \in \Lambda_0} \sum_{\psi \in b_\lambda} \overline{e_\lambda(\Phi_\psi)(x)} l'_{\sigma(\lambda)}(J_\lambda^{*-1}(\psi))(x').$$

Remark 2.4 [2, Remark 2.11] If for all $\lambda \in \Lambda_0$ (respectively Λ_0') G_λ (respectively G_λ') has an MN-structure with respect to $C \cap G_\lambda$ and B_λ (respectively $C' \cap G_\lambda'$ and B_λ') and G_λ and G_λ' satisfy parts (1) and (2) of Theorem 2.3 with respect to $C \cap G_\lambda$, B_λ and $C' \cap G_\lambda'$, B_λ' then part (3) is automatically satisfied.

We set $\overline{C} = G \backslash C$ and $\overline{C}' = G' \backslash C'$.

Definition 2.5 [2, Proposition 2.15] An isometry $I : \mathbb{C}\,\mathrm{Irr}(B) \to \mathbb{C}\,\mathrm{Irr}(B')$ is called a generalized perfect isometry if $I(\mathbb{Z}\,\mathrm{Irr}(B)) = \mathbb{Z}\,\mathrm{Irr}(B')$ and if $\hat{I}(x, x') = 0$ for all $(x, x') \in (C \times \overline{C}') \cup (\overline{C} \times C')$.

Lemma 2.6 *Suppose all the hypotheses of Theorem 2.3 hold then* $I_{\{1\}}$ *is a generalized perfect isometry.*

Now let C (respectively C') be the set of p-regular elements of G (respectively G'). Also let $(K, \mathcal{R}, k)$ be a splitting p-modular system for all the groups considered in the rest of this paper. In particular for both G and G'.

Definition 2.7 We describe $I : \mathbb{C}\,\mathrm{Irr}(B) \to \mathbb{C}\,\mathrm{Irr}(B')$ as a Broué perfect isometry if

1. For every $(x, x') \in G \times G'$, $\hat{I}(x, x') \in |C_G(x)|\mathcal{R} \cap |C_{G'}(x')|\mathcal{R}$.
2. I is a generalized perfect isometry.

Lemma 2.8 *Let* $x \in G$ *and suppose* $A \leq G_{x_S}$ *has* p'*-index with* $x_C \in A$ *and* $\Phi \downarrow_A$ *a virtual projective character of* A *for all* $\Phi \in \mathbb{Z}b^\vee$, *where* b *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}\,\mathrm{Irr}(B_{x_S})^{(C \cap G_{x_S})}$. *Then* $\hat{I}(x, x') \in |C_G(x)|\mathcal{R}$.

Proof See the proof of [2, Theorem 2.20]. □

2.2 Projective Representations and Inertial Subgroups

Theorem 2.9 *(Clifford correspondence) Let* G *be a finite group,* N *a normal subgroup of* G, χ *a character of* N *and* $I_G(\chi)$ *the inertial subgroup of* χ *in* G. *If* ψ *is an irreducible constituent of* $\chi \uparrow^{I_G(\chi)}$ *then* $\psi \uparrow^G$ *is irreducible. Furthermore, every irreducible character of* G *is of this form for a unique* G*-orbit of characters of* N.

Proof See [7, Theorem 6.11]. □

Remark 2.10 Note that by the transitivity of induction $I_G(\chi)$ can be replaced with any subgroup containing $I_G(\chi)$ in the above theorem.

Lemma 2.11 *Let* G *be a finite group,* H *a normal subgroup and* ρ *a representation of* G *over* $\mathbb{C}$. *Suppose* $\rho = \rho_1 \otimes \rho_2$, *where* ρ_1 *and* ρ_2 *are irreducible projective representations* *of G with $\rho_2(h) = 1$ for all $h \in H$ and $\rho_1 \downarrow_H$ an irreducible representation of H, then ρ is irreducible.*

Proof Let $M = M_1 \otimes M_2$ be the $\mathbb{C}G$-module afforded by ρ with ρ_1 affording M_1 and ρ_2 affording M_2. We can view M_1 as an irreducible $\mathbb{C}H$-submodule of M and as $\mathbb{C}H$-modules $M \cong M_1^{\oplus t}$, where $t = \dim(\rho_2)$. Let L be an irreducible $\mathbb{C}H$-submodule of M and $\{m_1, \dots, m_t\}$ a basis for M_2. By Schur's lemma each composition

$$M_1 \cong L \hookrightarrow M \twoheadrightarrow M_1 \otimes \mathbb{C}m_i \cong M_1$$

is given by a scalar and hence $L = M_1 \otimes \mathbb{C}m$ for some non-zero $m \in M_2$. Therefore any $\mathbb{C}H$-submodule of M is of the form $M_1 \otimes V$, for some vector subspace V of M_2. As ρ_2 is irreducible then any $\mathbb{C}G$-submodule of M containing $M_1 \otimes V$ must be the whole of M or $V = 0$. Hence ρ is irreducible. □

3 Combinatorics of Partitions

The combinatorics of partitions is very important when describing the characters of both the symmetric groups and their double covers. In this section we introduce all the combinatorics relevant to us in this paper.

For $n \in \mathbb{N}_0$ let $\mathcal{P}_n$ be the set of partitions of n. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_t > 0) \in \mathcal{P}_n$ then we define $l(\lambda) := t$ and $\sigma(\lambda) = (-1)^{n-t}$. For $\lambda \in \mathcal{P}_n$ and q a positive integer we take from [14] the notion of a q-hook and q-sign $\delta_q(\lambda)$. If b is a q-hook of λ such that the partition obtained by removing b is μ then we set $c^\lambda_\mu := b$ and $L(b)$ to be the leg length of b. Let $\lambda_{(q)}$, $(\lambda)^{(q)}$ and $w_q(\lambda)$ be the q-core, q-quotient and q-weight of λ respectively. Finally we set $M_q(\lambda)$ to be the set of partitions of $(n - q)$ obtained by removing a q-hook from λ.

Let $\mathcal{D}_n$ be the set of partitions of n with distinct parts and

$$\mathcal{D}^+_n := \{\lambda \in \mathcal{D}_n | \sigma(\lambda) = 1\}, \mathcal{D}^-_n := \{\lambda \in \mathcal{D}_n | \sigma(\lambda) = -1\}.$$

In addition we define

$$\mathcal{O}_n := \{\lambda \in \mathcal{P}_n | \text{all parts of } \lambda \text{ are odd}\}.$$

For $\lambda \in \mathcal{D}_n$ and q an odd positive integer we again take from [14] the notion of a q-bar and $\overline{q}$-sign $\delta_{\overline{q}}(\lambda)$. If b is a q-bar of λ such that the partition obtained by removing b is μ then we set $\overline{c}^\lambda_\mu = b$ and $L(b)$ to be the leg length of b. Let $\lambda_{(\overline{q})}$, $\lambda^{(\overline{q})}$ and $w_{\overline{q}}(\lambda)$ be the q-bar core, q-bar quotient and $\overline{q}$-weight of λ respectively. Finally we set $M_{\overline{q}}(\lambda)$ to be the set of partitions of $(n - q)$ obtained by removing a q-bar from λ.

Suppose $\lambda \in \mathcal{D}_n$ with $\overline{q}$-quotient $\lambda^{(\overline{q})} = (\lambda_0, \dots, \lambda_{(q-1)/2})$. Now removing an mq-bar b from λ, for some $m \in \mathbb{N}$, corresponds to removing an m-bar from λ_0 or an m-hook from λ_j for some $(1 \leq j \leq \lambda_{(q-1)/2})$. We denote this m-bar/hook by b' and by $L(b')$ its leg length as an m-bar/hook.

Lemma 3.1 *In the above setting*

$$(-1)^{L(b)} = (-1)^{L(b')}\delta_{\overline{q}}(\lambda)\delta_{\overline{q}}(\mu),$$

where μ is the partition obtained from λ by removing b.

Proof See [13, Corrollary 3.8]. □

4 Double Covers of Symmetric and Alternating Groups

In this section we introduce double covers of symmetric and alternating groups and gather together some basic facts about them.

Let n be a positive integer. We define the symmetric group on n letters,

$$S_n := \langle s_1, \dots, s_{n-1} \mid s_j^2 = 1, s_j s_k = s_k s_j \text{ if } |j-k| > 1, (s_j s_{j+1})^3 = 1 \text{ for } (0 \le j \le n-1)\rangle.$$

We now label the irreducible chraracters of S_n. For more details see [8, Theorem 2.1.11].

Theorem 4.1 *The irreducible characters of S_n are labeled by partitions of n.*

For $\lambda \vdash n$ we label by Sym_λ the irreducible character of S_n corresponding to λ through the above theorem. If $\lambda = (\lambda_1, \dots, \lambda_t)$ with $\lambda_j \vdash n_j$ then λ labels a character of $S_{n_1} \times \cdots \times S_{n_t}$ that we label Sym_λ.

We also define the double covers S_n^+ and S_n^- of S_n,

$$\begin{aligned}
S_n^+ := \langle z, t_1, \dots, t_{n-1} \mid z^2 = 1, t_j z = z t_j, t_j^2 = 1,\\
t_j t_k = z t_k t_j \text{ if } |j-k| > 1, (t_j t_{j+1})^3 = z \text{ for } (0 \le j \le n-1)\rangle,\\
S_n^- := \langle z, t_1, \dots, t_{n-1} \mid z^2 = 1, t_j z = z t_j, t_j^2 = z,\\
t_j t_k = z t_k t_j \text{ if } |j-k| > 1, (t_j t_{j+1})^3 = 1 \text{ for } (0 \le j \le n-1)\rangle.
\end{aligned}$$

From now on we will use $\tilde{S}_n$ to denote either S_n^+ or S_n^-. Note we have a homomorphism

$$\begin{aligned}
\theta_n : \tilde{S}_n &\to S_n\\
t_j &\mapsto s_j
\end{aligned}$$

with kernel $\{1, z\}$. We define $\tilde{A}_n$ to be $\theta_n^{-1}(A_n)$.

The conjugacy classes of S_n are labelled by partitions of n corresponding to the cycle type of the elements in that conjugacy class. If $\pi \vdash n$ and C_π is the corresponding conjugacy class of S_n then $\tilde{C}_\pi = \theta_n^{-1}(C_\pi)$ is the union of one or two conjugacy classes of $\tilde{S}_n$. If $\pi = (1^{\pi_1}, 2^{\pi_2}, \dots)$ and $g \in C_\pi$ then $C_{S_n}(g) \cong \prod_j (C_j \wr S_{\pi_j})$ and so $|C_{S_n}(g)| = \prod_j j^{\pi_j}\pi_j!$. From the above statement about conjugacy classes we can deduce that for $g \in \tilde{S}_n$

$$|C_{\tilde{S}_n}(g)| = |C_{S_n}(\theta_n(g))| \text{ or } 2|C_{S_n}(\theta_n(g))|.$$

We therefore have the following lemma.

Lemma 4.2 *Let $g \in \tilde{S}_n$ be in the conjugacy class $\tilde{C}_\pi$ for some $\pi = (1^{\pi_1}, 2^{\pi_2}, \dots) \vdash n$. Then*

$$|C_{\tilde{S}_n}(g)| = \prod_j j^{\pi_j}\pi_j! \text{ or } 2\prod_j j^{\pi_j}\pi_j!.$$

If $g \in S_n$ is in the conjugacy class C_π with $\pi \in \mathcal{O}_n$. Then g has odd order say m. Now let $h \in \tilde{S}_n$ with $\theta_n(h) = g$ and so $h^m \in \{1, z\}$. By multiplying by z we can assume h has order m and we use $o(g)$ to denote such an h.

For any subgroup H of $\tilde{S}_n$ with $z \in H$ we call an irreducible character χ of H an irreducible spin character if $z \notin \ker(\chi)$ and denote the set of such characters by $\mathrm{IrrSp}(H)$. If in addition $H \not\leq \tilde{A}_n$ then an irreducible spin character χ of H is referred to as self-associate if $\epsilon.\chi = \chi$, where ϵ is the sign character of H with kernel $H \cap \tilde{A}_n$. When the embedding of H in an $\tilde{S}_n$ is clear from the context then ϵ will always have this meaning. We denote by $\mathrm{IrrSp}^+(H)$ and $\mathrm{IrrSp}^-(H)$ the set of self-associate and non-self-associate characters of H respectively.

If $\chi \in \mathrm{IrrSp}^+(H)$ then $\chi \downarrow_{H\cap\tilde{A}_n}$ is the sum of two distinct irreducible spin characters of $H \cap \tilde{A}_n$. We label these two characters $\overline{\chi}^+, \overline{\chi}^-$. If $\chi \in \mathrm{IrrSp}^-(H)$ then $\chi \downarrow_{H\cap\tilde{A}_n} = (\epsilon.\chi) \downarrow_{H\cap\tilde{A}_n}$ is an irreducible spin character of $H \cap \tilde{A}_n$. We label this character $\overline{\chi}$.

Schur proved in [16] that there is the following labelling of the irreducible spin characters of $\tilde{S}_n$ and we adopt Schur's labelling for the rest of this paper.

Theorem 4.3 *The irreducible spin characters of $\tilde{S}_n$ are labelled in the following way. Each $\lambda \in \mathcal{D}^+$ labels an irreducible self-associate spin character of $\tilde{S}_n$. We denote such a character by ξ_λ. Each $\lambda \in \mathcal{D}^-$ labels an associate pair of irreducible spin characters of $\tilde{S}_n$. We denote such a pair by $\xi_\lambda^+, \xi_\lambda^-$. Furthermore, the above characters form a complete list of irreducible spin characters of $\tilde{S}_n$.*

Due to the remarks preceeding the above theorem we can also label the irreducible spin characters of $\tilde{A}_n$ and we make the choice of labelling as described in [2, §4].

Theorem 4.4 [16] *Let $\lambda \in \mathcal{D}_n$, $\pi \in \mathcal{P}_n$ and $g \in \tilde{C}_\pi$.*

1. *If $\sigma(\lambda) = 1$ then $\xi_\lambda(g) \neq 0$ only if $\pi \in \mathcal{O}_n$.*
2. *If $\sigma(\lambda) = -1$ then $\xi_\lambda^\pm(g) \neq 0$ only if $\pi \in \mathcal{O}_n$ or $\pi = \lambda$. Furthermore if $\pi = \lambda$ then*

$$\xi_\lambda^\pm(g) = \pm i^{(n-l(\lambda)+1)/2}\frac{\sqrt{(\lambda_1\lambda_2\dots)}}{2}.$$

Now suppose $\sigma(\pi) = 1$ and so $g \in \tilde{A}_n$.

3. *If $\sigma(\lambda) = 1$ then $\overline{\xi}_\lambda^+(g) \neq \overline{\xi}_\lambda^-(g)$ only if $\pi = \lambda$ and in this case*

$$\overline{\xi}_\lambda^+(g) - \overline{\xi}_\lambda^-(g) = \pm i^{(n-l(\lambda))/2}\sqrt{(\lambda_1\lambda_2\dots)}.$$

We conclude this section by discussing the p-blocks of $\tilde{S}_n$ and $\tilde{A}_n$, where p is an odd prime.

Theorem 4.5 *Let $\lambda, \mu \in \mathcal{D}_n$. If $w_{\overline{p}}(\lambda) = 0$ and $\sigma(\lambda) = -1$ then ξ_λ^+ and ξ_λ^- lie in p-blocks of $\tilde{S}_n$ on their own, otherwise $\xi_\lambda^{(\pm)}$ and $\xi_\mu^{(\pm)}$ lie in the same p-block if and only if $\lambda_{(\overline{p})} = \mu_{(\overline{p})}$.*

Proof See [6, Theorem 1.1]. □

If γ is a p-bar core then we denote by $\tilde{S}_{n,\gamma}$ the block of $\tilde{S}_n$ corresponding to γ or by $\tilde{S}_{n,\gamma}^{\pm}$ the block of $\tilde{S}_n$ containing $\xi_\gamma^{\pm}$ if $\gamma \in \mathcal{D}_n^-$. We label the corresponding block idempotent $e_{n,\gamma}^{(\pm)}$.

Theorem 4.6 *Let $w \in \mathbb{N}_0$, $n \geq pw$ and let $\gamma \vdash (n - pw)$ be a p-bar core. Consider the subgroup*

$$S_{pw} \leq S_{n-pw} \times S_{pw} \leq S_n.$$

Any Sylow p-subgroup P of $\theta_n^{-1}(S_{pw}) \leq \tilde{S}_n$ is a defect group of $\tilde{S}_{n,\gamma}^{(\pm)}$. Furthermore,

$$N_{\tilde{S}_n}(P) = \tilde{S}_{n-pw} N_{\tilde{S}_{pw}}(P)$$

and if $w > 0$ then the idempotent of the Brauer correspondent of $\tilde{S}_{n,\gamma}$ is

$$\begin{cases} e_{n-pw,\gamma} & \text{if } \sigma(\gamma) = 1, \\ e_{n-pw,\gamma}^{+} + e_{n-pw,\gamma}^{-} & \text{if } \sigma(\gamma) = -1. \end{cases}$$

Proof See [3, Theorems A and B and Corrollary 26]. □

Using clifford theory we can deduce the corresponding theorems for $\tilde{A}_n$. For proofs see [9, Proposition 3.16].

Theorem 4.7 *Let $\lambda, \mu \in \mathcal{D}_n$. If $w_{\overline{p}}(\lambda) = 0$ and $\sigma(\lambda) = 1$ then $\overline{\xi}_\lambda^+$ and $\overline{\xi}_\lambda^-$ lie in p-blocks of $\tilde{A}_n$ on their own, otherwise $\overline{\xi}_\lambda^{(\pm)}$ and $\overline{\xi}_\mu^{(\pm)}$ lie in the same p-block if and only if $\lambda_{(\overline{p})} = \mu_{(\overline{p})}$.*

Let $w \in \mathbb{N}_0$ and $\gamma \vdash (n-pw)$ be a p-bar core. Then we label the corresponding block(s) of $\tilde{A}_n$ by $\tilde{A}_{n,\gamma}^{(\pm)}$ and by $\overline{e}_\gamma^{(\pm)}$ the corresponding block idempotent of $\tilde{A}_{n,\gamma}^{(\pm)}$.

Theorem 4.8

1. *If $w = 0$ and $\sigma(\lambda) = -1$ then $\overline{e}_{n,\gamma} = e_{n,\gamma}^+ + e_{n,\gamma}^-$.*
2. *If $w = 0$ and $\sigma(\lambda) = 1$ then $e_{n,\gamma} = \overline{e}_{n,\gamma}^+ + \overline{e}_{n,\gamma}^-$.*
3. *If $w > 0$ then $e_{n,\gamma} = \overline{e}_{n,\gamma}$.*

Theorem 4.9 *Any Sylow p-subgroup P of $\theta_n^{-1}(S_{pw}) \leq \tilde{S}_n$ is a defect group of $\tilde{A}_{n,\gamma}^{(\pm)}$. Furthermore,*

$$N_{\tilde{A}_n}(P) = \tilde{S}_{n-pw} N_{\tilde{S}_{pw}}(P) \cap \tilde{A}_n$$

and the idempotent of the Brauer correspondent of $\tilde{A}_{n,\gamma}^{(\pm)}$ is

$$\begin{cases} \overline{e}_{n-pw,\gamma} & \text{if } \sigma(\gamma) = -1, \\ \overline{e}_{n-pw,\gamma}^{+} + \overline{e}_{n-pw,\gamma}^{-} & \text{if } \sigma(\gamma) = 1. \end{cases}$$

5 Clifford Algebras

In this section we introduce Clifford algebras for the purpose of later constructing the characters of parabolic subgroups of the double covers. For more details see [17, §3]. For any positive integer n we define the Clifford algebra $\mathcal{C}_n$ to be the $\mathbb{C}$-algebra generated by $e_1, \dots, e_n$ subject to the relations $e_j^2 = 1$ and $e_j e_k = -e_k e_j$ if $j \neq k$. In particular $\mathcal{C}_n$ has $\mathbb{C}$-basis $\{e_I\}_I$ where I runs over the subsets of $[n] := \{1, \dots, n\}$ and $e_I := e_{j_1} \cdot \dots \cdot e_{j_t}$ if $I = \{j_1 < \dots < j_t\}$ and $e_\varnothing := 1$. We also want to define the special Clifford algebra $\mathcal{C}_n^+$. We define this to be the subalgebra of $\mathcal{C}_n$ with $\mathbb{C}$-basis $\{e_I\}_I$ where I runs over the subsets of $[n]$ of even size.

Lemma 5.1

1. *If $n = 2k$ is even then, as $\mathbb{C}$-algebras, $\mathcal{C}_n \cong M_{2^k}(\mathbb{C})$, the algebra of $2^k \times 2^k$ matrices over $\mathbb{C}$. The character of this representation is given by $\chi_n^{\mathcal{C}} : \sum_I c_I e_I \mapsto 2^k c_\varnothing$.*
2. *If $n = 2k+1$ is odd then, as $\mathbb{C}$-algebras, $\mathcal{C}_n \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$. Futhermore, this isomorphism induces two irreducible characters of $\mathcal{C}_n$ given by $\chi_n^{\mathcal{C}\pm} : \sum_I c_I e_I \mapsto 2^k c_\varnothing \pm (2i)^k c_{[n]}$.*
3. *If $n = 2k$ is even then as $\mathbb{C}$-algebras, $\mathcal{C}_n^+ \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$. Futhermore, this isomorphism induces two irreducible characters of $\mathcal{C}_n$ given by $\chi_n^{\mathcal{C}\pm} : \sum_I c_I e_I \mapsto 2^{k-1} c_\varnothing \pm i(2i)^{k-1} c_{[n]}$.*
4. *If $n = 2k+1$ is odd then, as $\mathbb{C}$-algebras, $\mathcal{C}_n^+ \cong M_{2^k}(\mathbb{C})$ and the character associated to this isomorphism is given by $\chi_n^{\mathcal{C}} : \sum_I c_I e_I \mapsto 2^k c_\varnothing$.*

Proof

1,2. See [17, §3].
3,4. We get the desired results through the isomorphism

$$\begin{aligned} \mathcal{C}_{n-1} &\to \mathcal{C}_n^+ \\ e_j &\mapsto i e_j e_n. \end{aligned}$$

$\square$

Lemma 5.2

1. *One can realise S_n^+ and S_n^- as subgroups of $\mathcal{C}_n$ via*

$$\begin{aligned} \phi_n^+ : S_n^+ &\to \mathcal{C}_n \\ t_j &\mapsto \tfrac{1}{\sqrt{2}}(e_j + e_{j+1}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_n^- : S_n^- &\to \mathcal{C}_n \\ t_j &\mapsto \tfrac{i}{\sqrt{2}}(e_j + e_{j+1}). \end{aligned}$$

2. *For* $\phi_n = \phi_n^{\pm}$ *we have*

$$\phi_n(t_j)e_j\phi_n(t_j^{-1}) = e_{j+1}$$

Proof

1. See [17, §3].
2. We treat only the case of ϕ_n^+, as the case of ϕ_n^- is similar. Note that $t_j^{-1} = t_j$ and so

$$\phi_n(t_j)e_j\phi_n(t_j^{-1}) = \frac{1}{\sqrt{2}}(e_j + e_{j+1})e_j\frac{1}{\sqrt{2}}(e_j + e_{j+1}) = \frac{1}{2}(e_j + e_{j+1} + e_{j+1} - e_j) = e_{j+1}.$$

□

6 Characters of Parabolic Subgroups

Let $H \leq \tilde{S}_n$ with $z \in H$ and $H \nleq \tilde{A}_n$. If V is the representation space of a self-associate irreducible spin representation ρ of H then there exists some $\mathcal{S} : V \to V$ such that $\mathcal{S}\rho(h)\mathcal{S}^{-1} = \epsilon(h)\rho(h)$ for all $h \in H$. Note that by Schur's lemma $\mathcal{S}^2 = \lambda\,\mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$. We can always scale $\mathcal{S}$ so that $\lambda = 1$. In this case we call $\mathcal{S}$ an associator and $\mathcal{S}$ is uniquely determined up to multiplication by ± 1.

Let $t \in \mathbb{N}$, $n_1, \ldots, n_t \in \mathbb{N}$ and $n := \sum_j n_j$. Suppose we have a family of groups H_j with $H_j \leq \tilde{S}_{n_j}$ and $H_j \nleq \tilde{A}_{n_j}$ for $(1 \leq j \leq t)$. Now we explicitly construct the irreducible spin characters of

$$H_1 \ldots H_t \leq \tilde{S}_{n_1} \ldots \tilde{S}_{n_t} \leq \tilde{S}_n.$$

For this construction we follow [17, §4], where it is assumed each $H_j = \tilde{S}_{n_j}$. However, the proofs run through with no extra complications in this more general setup.

Let $\chi_j \in \mathrm{IrrSp}(H_j)$ and assume without loss of generality that $\chi_1, \ldots, \chi_r$ are all self-associate and $\chi_{r+1}, \ldots, \chi_t$ are all non-self-associate. Let V_j be the representation space of the character χ_j and $\mathcal{S}_j$ an associator map of V_j for $(1 \leq j \leq r)$ and Id_{V_j} for $(r+1 \leq j \leq t)$. Finally let V be an irreducible representation space of $\mathcal{C}_{t-r}$. If $x \in H_j$ then we define the action of x on $V \otimes V_1 \otimes \cdots \otimes V_t$ as follows:

$$x.(v \otimes v_1 \otimes \cdots \otimes v_t) = \begin{cases} e_j.v \otimes A_1.v_1 \otimes \cdots \otimes A_t.v_t & \text{if } j > r \text{ and } x \in H_j \backslash \tilde{A}_{n_j}, \\ v \otimes A_1.v_1 \otimes \cdots \otimes A_t.v_t & \text{otherwise,} \end{cases} \tag{1}$$

where

$$(A_1, \ldots, A_t) = \begin{cases} (1, \ldots, 1, x, 1, \ldots, 1) & \text{if } x \in H_j \cap \tilde{A}_{n_j}, \\ (\mathcal{S}_1, \ldots, \mathcal{S}_{j-1}, x, 1, \ldots, 1) & \text{if } x \in H_j \backslash \tilde{A}_{n_j}. \end{cases}$$

(Note that in both cases $A_j = x$.) This defines a representation of $H_1 \ldots H_t$ and it is non-self-associate if and only if $(t - r)$ is odd. Furthermore, one can obtain its associate by replacing V with the other irreducible representation of $\mathcal{C}_{t-r}$ or by replacing χ_j with $\epsilon.\chi_j$ for some $j > r$.

Now suppose that for each j with $(1 \leq j \leq t)$ we have sets Λ_j^+ and Λ_j^- with

$$\mathrm{IrrSp}^+(H_j) = \left\{\chi_\lambda | \lambda \in \Lambda_j^+\right\}, \mathrm{IrrSp}^-(H_j) = \left\{\chi_\lambda^{\pm} | \lambda \in \Lambda_j^-\right\}.$$

For $(1 \leq j \leq t)$, write $\Lambda_j = \Lambda_j^+ \cup \Lambda_j^-$ and let $\lambda_j \in \Lambda_j$. Then we denote by $\chi_{(\lambda_1,\ldots,\lambda_t)}$ the corresponding character of $H_1 \ldots H_t$ if $\lambda_j \in \Lambda_j^-$ for an even number of j's or by $\chi^{\pm}_{(\lambda_1,\ldots,\lambda_t)}$ if $\lambda_j \in \Lambda_j^-$ for an odd number of j's.

Theorem 6.1 *A complete set of irreducible spin characters of $H_1 \ldots H_t$ are given by*

$$\{\chi_{(\lambda_1,\ldots,\lambda_t)} | \lambda_j \in \Lambda_j^- \textit{ for an even number of } j\textit{'s}\} \cup \{\chi^{\pm}_{(\lambda_1,\ldots,\lambda_t)} | \lambda_j \in \Lambda_j^- \textit{ for an odd number of } j\textit{'s}\}.$$

Proof See [17, 4.3]. □

We can explicitly write down the irreducible spin characters when $t = 2$.

Lemma 6.2 *Let $s_j \in H_j$ for $j = 1, 2$.*

1. *If $\lambda_1 \in \Lambda_1^+, \lambda_2 \in \Lambda_2^+$ then*

$$\chi_{(\lambda_1,\lambda_2)}(s_1 s_2) = \chi_{\lambda_1}(s_1)\chi_{\lambda_2}(s_2).$$

2. *If $\lambda_1 \in \Lambda_1^+, \lambda_2 \in \Lambda_2^-$ then*

$$\chi^{\pm}_{(\lambda_1,\lambda_2)}(s_1 s_2) = \begin{cases} \overline{\chi}^{\pm}_{\lambda_1}(s_1)\chi^{\pm}_{\lambda_2}(s_2) + \overline{\chi}^{\mp}_{\lambda_1}(s_1)\chi^{\mp}_{\lambda_2}(s_2) & \textit{if } s_1 \in \tilde{A}_{n_1}, \\ 0 & \textit{otherwise.} \end{cases}$$

3. *If If $\lambda_1 \in \Lambda_1^-, \lambda_2 \in \Lambda_2^-$ then*

$$\chi_{(\lambda_1,\lambda_2)}(s_1 s_2) = \begin{cases} \chi^{+}_{\lambda_1}(s_1)\overline{\chi}_{\lambda_2}(s_2) + \chi^{-}_{\lambda_1}(s_1)\overline{\chi}_{\lambda_2}(s_2) & \textit{if } s_2 \in \tilde{A}_{n_2}, \\ 0 & \textit{otherwise.} \end{cases}$$

Note that in (1) we are also setting up a labelling of characters.

Proof See [17, 4.2]. □

We would also like to describe the irreducible spin characters of $H_1.H_2 \cap \tilde{A}_n$.

Lemma 6.3 *Let $s_j \in H_j$ for $j = 1, 2$ with $s_1 s_2 \in H_1.H_2 \cap \tilde{A}_n$.*

1. *If $\lambda_1 \in \Lambda_1^+, \lambda_2 \in \Lambda_2^+$ then*

$$\overline{\chi}^{\pm}_{(\lambda_1,\lambda_2)}(s_1 s_2) = \begin{cases} \overline{\chi}^{+}_{\lambda_1}(s_1)\overline{\chi}^{\pm}_{\lambda_2}(s_2) + \overline{\chi}^{-}_{\lambda_1}(s_1)\overline{\chi}^{\mp}_{\lambda_2}(s_2) & \textit{if } s_1 \in \tilde{A}_{n_1} \textit{ and } s_2 \in \tilde{A}_{n_2}, \\ 0 & \textit{otherwise.} \end{cases}$$

2. *If $\lambda_1 \in \Lambda^+, \lambda_2 \in \Lambda^-$ then*

$$\overline{\chi}_{(\lambda_1,\lambda_2)}(s_1 s_2) = \begin{cases} \chi_{\lambda_1}(s_1)\chi^{\pm}_{\lambda_2}(s_2) & \textit{if } s_1 \in \tilde{A}_{n_1} \textit{ and } s_2 \in \tilde{A}_{n_2}, \\ 0 & \textit{otherwise.} \end{cases}$$

3. *If $\lambda_1 \in \Lambda^-, \lambda_2 \in \Lambda^-$ then*

$$\overline{\chi}^{\pm}_{(\lambda_1,\lambda_2)}(s_1 s_2) = \begin{cases} \chi^{+}_{\lambda_1}(s_1)\chi^{+}_{\lambda_2}(s_2) & \textit{if } s_1 \in \tilde{A}_{n_1} \textit{ and } s_2 \in \tilde{A}_{n_2}, \\ \pm i\chi^{+}_{\lambda_1}(s_1)\chi^{+}_{\lambda_2}(s_2) & \textit{otherwise.} \end{cases}$$

Again note that in 1 and 3 we are setting up a labelling of characters.

Proof We can do this by directly decomposing the vector space in (1).

1. Let $V_j \cong V_j^+ \oplus V_j^-$ be a decomposition of V_j as a $\mathbb{C}(H_j \cap \tilde{A}_{n_j})$-module for $j = 1, 2$. Then

$$V_1 \otimes V_2 \cong (V_1^+ \otimes V_2^+ \oplus V_1^- \otimes V_2^-) \oplus (V_1^+ \otimes V_2^- \oplus V_1^- \otimes V_2^+)$$

is a decomposition of $V_1 \otimes V_2$ as a $\mathbb{C}(H_1.H_2 \cap \tilde{A}_n)$-module. The result follows.
2. If $\lambda_1 \in \Lambda^+, \lambda_2 \in \Lambda^-$ then $\chi^{\pm}_{(\lambda_1,\lambda_2)} \downarrow_{H_1.H_2 \cap \tilde{A}_n}$ is irreducible.
3. If $\lambda_1 \in \Lambda^-, \lambda_2 \in \Lambda^-$ then let $V \cong V^+ \oplus V^-$ be a decomposition of V in (1) as a $\mathcal{C}_2^+$-module. Then

$$V \otimes V_1 \otimes V_2 \cong (V^+ \otimes V_1 \otimes V_2) \oplus (V^+ \otimes V_1 \otimes V_2)$$

is a decomposition of $V \otimes V_1 \otimes V_2$ as a $\mathbb{C}(H_1.H_2 \cap \tilde{A}_n)$-module. The result follows.

□

7 $\tilde{N}_p^t \tilde{S}_t$ and its Characters

For the rest of the paper we fix an odd prime p. We now introduce the group $\tilde{N}_p^t \tilde{S}_t$. The study of this group, its characters and its MN-structure will form the bulk of the rest of the paper. We will eventually use this group to construct intermediate Broué perfect isometries in order to prove Theorem 1.1.

7.1 Introduction to $\tilde{N}_p^t \tilde{S}_t$

Consider the subgroup $N_p := N_{S_p}(C_p) \leq S_p$ where C_p is generated by a p-cycle. Note that $N_p \cong C_p \rtimes C_{p-1}$ where C_{p-1} acts as the full automorphism group of C_p. We denote by $\tilde{N}_p$ the subgroup $\theta_p^{-1}(N_p)$ of $\tilde{S}_p$. Note that $\tilde{N}_p \cong C_p \rtimes \theta_p^{-1}(C_{p-1})$.

Consider the subgroup $N_p^t \leq S_p^t \leq S_{pt}$ for some positive integer t. We denote by $\tilde{N}_p^t$ the subgroup $\theta_{pt}^{-1}(N_p^t) \leq \theta_{pt}^{-1}(S_p^t) \leq \tilde{S}_{pt}$ and by $\tilde{N}_p^t \tilde{S}_t$ the subgroup $\theta_{pt}^{-1}(N_p^t \rtimes S_t)$ of $\theta_{pt}^{-1}(S_p^t \rtimes S_t) \leq \tilde{S}_{pt}$, where S_t acts by permuting the factors of N_p^t. We will constantly view $\tilde{N}_p^t \tilde{S}_t$ as a subgroup of $\tilde{S}_{pt}$. By [10, Lemma 3.5] we have that $\theta_{pt}^{-1}(S_t) \cong S_t^{\pm}$. In either case we will denote $\theta_{pt}^{-1}(S_t)$ by $\tilde{S}_t$. For any pair of subgroups $A \leq \tilde{N}_p^t$, $H \leq \tilde{S}_t$ we denote by AH the subgroup of $\tilde{N}_p^t \tilde{S}_t$ generated by A and H.

Let $(1 \leq j < l \leq t)$. We denote by $[j \to l]$ the isomorphism between the j^{th} and $l^{\text{th}} factor of \tilde{N}_p^t$ given by conjugating by some $s \in \tilde{S}_t$, where $\theta_t(s)$=$(j, j+1, \dots, l-1, l)$. Note that by considering the case $l = j + 1$ and induction, one has

$$[l \to m] \circ [j \to l] = [j \to m] \text{ for } (1 \leq j < l < m \leq t).$$

Now for some positive integer $q \leq t$ consider the subgroup of $\tilde{N}_p^t \tilde{S}_t$ that fixes $\{1, \dots, pq\}$ where $\tilde{N}_p^t \tilde{S}_t$ is viewed as a subgroup of $\tilde{S}_{pt}$. We denote this subgroup by $\tilde{N}_p^{t-q} \tilde{S}_{t-q}[q]$ and set

$$\begin{aligned} \tilde{N}_p^{t-q}[q] &:= \tilde{N}_p^{t-q} \tilde{S}_{t-q}[q] \cap \tilde{N}_p^t, \\ \tilde{S}_{t-q}[q] &:= \tilde{N}_p^{t-q} \tilde{S}_{t-q}[q] \cap \tilde{S}_t. \end{aligned}$$

Lemma 7.1 *There exists an isomorphism* $[q]$ *between* $\tilde{N}_p^{t-q}\tilde{S}_{t-q}$ *and* $\tilde{N}_p^{t-q}\tilde{S}_{t-q}[q]$ *given by*

$$\begin{aligned}[q]: \tilde{N}_p^{t-q}\tilde{S}_{t-q} &\to \tilde{N}_p^{t-q}\tilde{S}_{t-q}[q] \\ x &\mapsto [j \to j+q](x) \quad \textit{for } x \textit{ in the } j^{th} \textit{ factor of } \tilde{N}_p^{t-q} \textit{ for } (1 \le j \le t-q) \\ t_j &\mapsto t_{j+q} \qquad\qquad\quad \textit{for } (1 \le j \le t-q).\end{aligned}$$

Proof It is easy to see that $[q]$ induces an isomorphism between $\tilde{N}_p^{t-q}$ and $\tilde{N}_p^{t-q}[q]$ and also between $\tilde{S}_{t-q}$ and $\tilde{S}_{t-q}[q]$. Let $(1 \le j, l \le t)$ and x be in the j^{th} factor of $\tilde{N}_p^{t-q}$. Then if $l \ne j, j+1$,

$$[q](t_l x t_l^{-1}) = [q](z^{(1-\epsilon(x))/2}x) = s(z^{(1-\epsilon(x))/2}x)s^{-1},$$

where $\theta_t(s) = (j, j+1, \dots, j+q-1, j+q)$. On the other hand

$$[q](t_l)[q](x)[q](t_l^{-1}) = t_{l+q}sxs^{-1}t_{l+q}^{-1} = z^{(1-\epsilon(x))/2}sxs^{-1}.$$

Now if $l = j$ then

$$[q](t_l x t_l^{-1}) = s' t_l x t_l^{-1} s'^{-1},$$

where $\theta_t(s') = (j+1, \dots, j+q-1, j+q+1)$. Therefore $\theta_t(s't_l) = (j, j+1, \dots, j+q, j+q+1)$

$$[q](t_l)[q](x)[q](t_l^{-1}) = t_{l+q}sxs^{-1}t_{l+q}^{-1} = (t_{l+q}s)x(t_{l+q}s)^{-1},$$

and $\theta_t(t_{l+q}s) = (j, j+1, \dots, j+q, j+q+1)$. $\square$

Let y_0 be a generator of C_p and y_1 a generator of C_{p-1} then

$$y_0, y_1, y_2 := y_1^2, \dots, y_{p-1} := y_1^{p-1}$$

is a complete list of representatives of conjugacy classes of N_p. We write

$$N_p \wr S_t = ((N_p)_1 \times \cdots \times (N_p)_t) \rtimes S_t.$$

Let $n_1, \dots, n_l \le t$ be distinct positive integers and $x_j \in (N_p)_{n_j}$ for $(1 \le j \le l)$. We denote by

$$x := ((x_1, \dots, x_l); (n_1, \dots, n_l))$$

the element $(x_1, \dots, x_l).(n_1, \dots, n_l)$ of $N_p \wr S_t$, where $(x_1, \dots, x_l)$ is understood to be in $(N_p)_{n_1} \times \cdots \times (N_p)_{n_l}$ and $(n_1, \dots, n_l)$ is the usual notation for a cycle in S_t. We describe x as a cycle of length l in $N_p \wr S_t$ and associate to it the element $f(x) := (x_1 \cdot \ldots \cdot x_l) \in N_p$. Now let $g = \prod x \in N_p \wr S_t$ be a product of disjoint cycles of $N_p \wr S_t$, where disjoint means the corresponding elements of S_t are disjoint cycles. We associate to g the multipartition $\pi = (\pi_0, \dots, \pi_{p-1})$ where π_j has as its parts the lengths of the cycles x with $f(x)$ conjugate to y_j in N_p. We say g is of type π. By [8, 4.2.8] the type of an element in $N_p \wr S_t$ completely determines its conjugacy class.

Now let $x_j \in \tilde{N}_p$ for $(1 \le j \le l)$ and $\tau \in \tilde{S}_t$ with $\theta_t(\tau) = (n_1, \dots, n_l)$. We denote by

$$x := ((x_1, \dots, x_l); \tau) \in \tilde{N}_p^t \tilde{S}_t$$

the element $(x_1 \dots x_l).\tau$ of $\tilde{N}_p^t \tilde{S}_t$ where it is understood that $x_j \in \theta_{pt}^{-1}((N_p)_{n_j}) \cong \tilde{N}_p$ for $(1 \le j \le l)$. We describe $g = \prod x \in \tilde{N}_p^t \tilde{S}_t$ as a product of disjoint cycles in $\tilde{N}_p^t \tilde{S}_t$ if $\theta_{pt}(g) = \prod \theta_{pt}(x)$ is a product of disjoint cycles in $N_p \wr S_t$. We also say g is of type π if

$\theta_{pt}(g)$ is of type π. Elements of a given type form either one or two conjugacy classes of $\tilde{N}_p^t\tilde{S}_t$.

If $x \in N_p \wr S_t$ then x is conjugate in $N_p \wr S_t$ to an element with disjoint cycle decomposition

$$\prod_j ((x_j, 1, \ldots, 1); \tau_j).$$

Then the order of $((x_j, 1, \ldots, 1); \tau_j)$ is $|\tau_j| \operatorname{ord}(x_j)$ and so

$$\operatorname{ord}(x) = \operatorname{ord}(\prod_j ((x_j, 1, \ldots, 1); \tau_j)) = \operatorname{lcm}_j(|\tau_j| \operatorname{ord}(x_j)).$$

Now for each $\tau_j = (n_{j,1}, \ldots, n_{j,l_j})$ set

$$C_j := \{(g, \ldots, g) | g \in C_{N_p}(x_j)\} \leq N_p^{l_j}.$$

Then

$$C_{N_p \wr S_t}(x) \cap N_p^t \cong \prod_j C_j.$$

Now let $x \in \tilde{N}_p^t\tilde{S}_t$. It is clear that

$$\operatorname{ord}(x) = \operatorname{ord}(\theta_{pt}(x)) \text{ or } 2 \operatorname{ord}(\theta_{pt}(x)).$$

Also, as in Section 4, we have

$$|C_{\tilde{N}_p^t\tilde{S}_t}(x)| = |C_{N_p \wr S_t}(\theta_{pt}(x))| \text{ or } 2|C_{N_p \wr S_t}(\theta_{pt}(x))|.$$

Bringing all the above together we have the following lemma:

Lemma 7.2 *Let $x \in \tilde{N}_p^t\tilde{S}_t$ have type π with $t < p$. Then*

1. *x is p-regular if and only if $\pi_0 = \varnothing$,*
2. $|C_{\tilde{N}_p^t\tilde{S}_t}(x)|_p = p^{l(\pi_0)+l(\pi_{p-1})}$.

Proof

1. This follows from the above comments.
2. This follows from the above comments, that C_p is self-centralising and that $p \nmid [N_p \wr S_t : N_p^t]$.

□

7.2 Characters of $\tilde{N}_p^t\tilde{S}_t$

There are p irreducible spin characters of $\tilde{N}_p$. The $(p-1)$ linear spin characters are given by inflating the $(p-1)$ linear spin characters of $\theta_p^{-1}(C_{p-1})$ and the remaining character is given by inducing any faithful linear character of $\theta_p^{-1}(C_p) \cong C_p \times \langle z \rangle$. All the linear characters are non-self-associate and the non-linear character is self associate. We label the $(p-1)/2$ associate pairs of linear characters ζ_j^+ and ζ_j^- for $(1 \leq i \leq (p-1)/2)$ and the self-associate character ζ_0.

Following the notation of Section 6 we denote by $\chi_t^{(\pm)}$ the character(s) of $\tilde{N}_p^t$ labelled by

$$(\zeta_0, \ldots, \zeta_0, \ldots, \zeta_{(p-1)/2}, \ldots, \zeta_{(p-1)/2}),$$

where $\boldsymbol{t} = (t_0, \dots, t_{(p-1)/2})$ with each ζ_j appearing t_j times. We want to describe the inertial subgroup of $\chi_{\boldsymbol{t}}^{(\pm)}$.

Lemma 7.3

$$I_{\tilde{N}_p^t \tilde{S}_t}(\chi_{\boldsymbol{t}}^{(\pm)})/\tilde{N}_p^t \cong \begin{cases} S_{t_0} \times S_{t_1} \times \cdots \times S_{t_{(p-1)/2}} & \textit{if } t - t_0 \textit{ is even,} \\ A_{t_0} \times S_{t_1} \times \cdots \times S_{t_{(p-1)/2}} & \textit{if } t - t_0 \textit{ is odd.} \end{cases}$$

Proof See [10, Proposition 3.13]. □

Adopting the notation of Lemma 7.1 we set

$$\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}} = \tilde{N}_p^t (\tilde{S}_{t_0}[t_0'] \tilde{S}_{t_1}[t_1'] \dots \tilde{S}_{t_{(p-3)/2}}[t'_{(p-3)/2}] \tilde{S}_{t_{(p-1)/2}}[t'_{(p-1)/2}]),$$

where $t_l' = \sum_{j=0}^{l-1} t_j$ for all $(0 \leq l \leq (p-1)/2)$.
By rearranging factors we can see that every irreducible spin character of $\tilde{N}_p^t$ is conjugate to $\chi_{\boldsymbol{t}}^{(\pm)}$ for a unique $\boldsymbol{t}$. However, when $(t - t_0)$ is odd we need to determine if $\chi_{\boldsymbol{t}}^{+}$ is conjugate to $\chi_{\boldsymbol{t}}^{-}$. In this case we have

$$[\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}} : I_{\tilde{N}_p^t \tilde{S}_t}(\chi_{\boldsymbol{t}}^{\pm})] = \begin{cases} 1 & \text{if } t_0 \leq 1, \\ 2 & \text{if } t_0 > 1. \end{cases}$$

Therefore we have an element that swaps $\chi_{\boldsymbol{t}}^{+}$ and $\chi_{\boldsymbol{t}}^{-}$ if and only if $t_0 > 1$ and hence we have the following lemma.

Lemma 7.4 *A complete list of representatives of $\tilde{N}_p^t \tilde{S}_t$-conjugacy classes of irreducible spin characters of $\tilde{N}_p^t$ are given by*

$$\{\chi_{\boldsymbol{t}} | \boldsymbol{t}, t - t_0 \textit{ even}\} \cup \{\chi_{\boldsymbol{t}}^{+} | \boldsymbol{t}, t - t_0 \textit{ odd, } t_0 > 1\} \cup \{\chi_{\boldsymbol{t}}^{\pm} | \boldsymbol{t}, t - t_0 \textit{ odd, } t_0 \leq 1\}.$$

With Theorem 2.9 and Remark 2.10 in mind we want to describe the irreducible constituents of $\chi_{\boldsymbol{t}}^{(\pm)} \uparrow^{\tilde{N}_p^t \tilde{S}_t}$. To do this let's first assume $t_0 = 0$.

Lemma 7.5 *If $t_0 = 0$ then $\chi_{\boldsymbol{t}}^{(\pm)}$ extends to a character of $\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}}$. Moreover, this character is self-associate if and only if t is even.*

Proof Using the description in Section 6 we can view the representation space of $\chi_{\boldsymbol{t}}^{(\pm)}$ as an irreducible $\mathcal{C}_t$-module. We can now define an action of $\tilde{S}_{\boldsymbol{t}}$ via the homomorphism

$$\tilde{S}_{\boldsymbol{t}} \hookrightarrow \tilde{S}_t \to \mathcal{C}_t,$$

where the second map is given by ϕ_t in Lemma 5.2. Note that by part (2) of this lemma and the identification of factors in Section 7.1 this does indeed define a representation of $\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}}$. The second part is immediate from 5.1. For more details see [10, Lemma 4.3]. □

We label the character(s) in the above lemma $\chi_{\tilde{\boldsymbol{t}}}^{(\pm)}$. Continuing with our assumption that $t_0 = 0$ we now are in a position to describe all the irreducible constituents of $\chi_{\boldsymbol{t}}^{(\pm)} \uparrow^{\tilde{N}_p^t \tilde{S}_t}$ in this case.

Lemma 7.6 *If $t_0 = 0$ then the irreducible constituents of $\chi_{\boldsymbol{t}}^{(\pm)} \uparrow^{\tilde{N}_p^t \tilde{S}_t}$ are precisely the characters of the form $\chi_{\bar{\boldsymbol{t}}}^{(\pm)} \otimes \chi$ where χ is an irreducible character of $S_{t_1} \times \cdots \times S_{t_{(p-1)/2}}$ inflated to $\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}}$. Moreover, $\chi_{\bar{\boldsymbol{t}}}^{(\pm)} \otimes \chi$ is self-associate if and only if t is even and if t is odd then $\chi_{\bar{\boldsymbol{t}}}^{\pm} \otimes \chi$ are associates.*

Proof For the first part we apply [7, Corollary 6.17].
For the second part we note that if t is even then by Lemma 7.5 $\chi_{\bar{\boldsymbol{t}}}^{(\pm)}(x) = 0$ and hence $(\chi_{\bar{\boldsymbol{t}}}^{(\pm)} \otimes \chi)(x) = 0$ for any $x \in \tilde{N}_p^t \tilde{S}_t \backslash \tilde{A}_{pt}$. If t is odd then there exists $x \in \tilde{N}_p^t \backslash \tilde{A}_{pt}$ such that $\chi_{\bar{\boldsymbol{t}}}^{(\pm)}(x) \neq 0$ and hence $(\chi_{\bar{\boldsymbol{t}}}^{(\pm)} \otimes \chi)(x) \neq 0$. It is clear that $\chi_{\bar{\boldsymbol{t}}}^{(\pm)} \otimes \chi$ are associates. $\square$

If $\lambda = (\varnothing, \lambda_1, \ldots, \lambda_{(p-1)/2})$ with $\lambda_j \vdash t_j$ then λ labels a character of $S_{t_1} \times \cdots \times S_{t_{(p-1)/2}}$ and we denote by χ_λ if t is even or by $\chi_\lambda^\pm$ if t is odd the corresponding character of $\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}}$. As described in Section 4 we also have the character(s) $\overline{\chi}_\lambda^{(\pm)}$ of $\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}} \cap \tilde{A}_{pw}$. We do not worry about the choices made for the labelling of $\chi_\lambda^\pm$ or $\overline{\chi}_\lambda^\pm$ only that it is fixed from now on.

We now turn our attention to the case $t_1 = \cdots = t_{(p-1)/2} = 0$. In this case, unless $t \leq 3$, $\chi_{\boldsymbol{t}}$ does not extend to a character of $\tilde{N}_p^t \tilde{S}_t$. However, we will show that it always extends to a character of $\tilde{N}_p^t \rtimes S_t$, where the action of $\tau \in S_t$ on $\tilde{N}_p^t$ is given by conjugating by an element of $\theta_t(\tau) \in \tilde{S}_t$. Let $\rho : \mathrm{GL}(U) \to \mathrm{GL}(U)$ be the representation corresponding to ζ_0 and $\mathcal{S} : U \to U$ an associator for ρ. Let U^+ (respectively U^-) be the 1-eigenspace (respectively (-1)-eigenspace) of $\mathcal{S}$ and let u_1 and u_2 be eigenvectors for S. We define the map

$$\begin{aligned} T : U \otimes U &\to U \otimes U \\ u_1 \otimes u_2 &\mapsto \eta(u_2 \otimes u_1), \end{aligned}$$

$$\text{where } \eta := \begin{cases} -1 & \text{if } u_1, u_2 \in U^-, \\ 1 & \text{otherwise,} \end{cases}$$

and extend linearly.

Lemma 7.7

1. *T commutes with $\mathcal{S} \otimes \mathcal{S}$.*
2. *$((\mathcal{S} \otimes T) \circ (T \otimes \mathcal{S}))^3 = 1$.*
3. *If $x \in \tilde{N}_p$ then $T \circ (\rho_1(x) \otimes 1) \circ T^{-1} = \mathcal{S}^{(1-\epsilon(x))/2} \otimes \rho_1(x)$.*

Proof

1. Let $u_1, u_2 \in U$ be eigenvalues of $\mathcal{S}$. Then,

$$(T \circ (\mathcal{S} \otimes \mathcal{S}))(u_1 \otimes u_2) = \eta(u_2 \otimes u_1) = ((\mathcal{S} \otimes \mathcal{S}) \circ T)(u_1 \otimes u_2),$$
$$\text{where } \eta := \begin{cases} 1 & \text{if } u_1, u_2 \in U^+, \\ -1 & \text{otherwise,} \end{cases}$$

and hence T and $\mathcal{S} \otimes \mathcal{S}$ commute.

2. Let $u_1, u_2, u_3 \in U$ be eigenvalues of $\mathcal{S}$. Then,
$$((\mathcal{S} \otimes T) \circ (T \otimes \mathcal{S}))(u_1 \otimes u_2 \otimes u_3) = \eta(u_2 \otimes u_3 \otimes u_1),$$
$$\text{where } \eta := \begin{cases} -1 & \text{if } u_1 \in U^+ \text{ and } u_2, u_3 \text{ do not have the same eigenvalue,} \\ 1 & \text{otherwise,} \end{cases}$$
and hence $((\mathcal{S} \otimes T) \circ (T \otimes \mathcal{S}))^3 = 1$.
3. Let $u_1, u_2 \in U$ be eigenvalues of $\mathcal{S}$. Then,
$$\begin{aligned}(T \circ (\rho_1(x) \otimes 1))(u_1 \otimes u_2) &= \eta u_2 \otimes u_1 = ((\mathcal{S}^{(1-\epsilon(x))/2} \otimes \rho_1(x)) \circ T)(u_1 \otimes u_2), \\ \text{where } \eta &:= \begin{cases} -1 & \text{if } u_1 \in U^+ \text{ and } u_2 \in U^-, \\ 1 & \text{otherwise,} \end{cases}\end{aligned}$$
and hence $T \circ (\rho_1(x) \otimes 1) \circ T^{-1} = \mathcal{S}^{(1-\epsilon(x))/2} \otimes \rho_1(x)$.

□

We decompose ζ_0 as a character of $\tilde{N}_p \cap \tilde{A}_p$ as $\zeta_0 = \overline{\zeta_0^+} + \overline{\zeta_0^-}$ where $\overline{\zeta_0^+}$ (respectively $\overline{\zeta_0^-}$) corresponds to the subspace U^+ (respectively U^-).

Now let $U_1, \dots, U_t$ be t isomorphic copies of U and let $\mathcal{S}_j$ be the linear map on U_j corresponding to $\mathcal{S}$ through the isomorphism with U. Now for $(1 \leq j \leq t-1)$ define the linear map
$$\begin{aligned} T_j : U_1 \otimes \cdots \otimes U_t &\to U_1 \otimes \cdots \otimes U_t \\ u_1 \otimes \cdots \otimes u_t &\mapsto \mathcal{S}_1(u_1) \otimes \cdots \otimes \mathcal{S}_{j-1}(u_{j-1}) \otimes T \otimes \mathcal{S}_{j+2}(u_{j+2}) \otimes \cdots \otimes \mathcal{S}_t(u_t). \end{aligned}$$
We now define a representation of $\tilde{N}_p^t \rtimes S_t$. Recall that $t_0 = t$ and therefore the representation space of $\chi_{\boldsymbol{t}}$ can be identified with $U_1 \otimes \cdots \otimes U_t$ via the description in Section 6. We denote this representation by $\rho_{\boldsymbol{t}}$.

Lemma 7.8 *The following action defines a representation ρ of $\tilde{N}_p^t \rtimes S_t$ on $U_1 \otimes \cdots \otimes U_t$ that extends $\rho_{\boldsymbol{t}}$,*
$$\rho(s_j)(u) := T_j(u) \textit{ for all } u \in U_1 \otimes \cdots \otimes U_t \textit{ and } (1 \leq j \leq t-1).$$

Proof One needs to check that ρ defines a representation of S_t and that $\rho(s_j x s_j^{-1}) = \rho(s_j)\rho(x)\rho(s_j^{-1})$ for all $(1 \leq j \leq t-1)$ and $x \in \tilde{N}_p^t$.

First we note that $T_j^2 = 1$ and, as T commutes with $\mathcal{S} \otimes \mathcal{S}$ by Lemma 7.7 part (1), we also have that $T_j T_l = T_l T_j$ for $|j-l| > 1$. Lemma 7.7 part (2) says that $((\mathcal{S} \otimes T) \circ (T \otimes \mathcal{S}))^3 = 1$ and hence that $(T_j T_{j+1})^3 = 1$ and so ρ does indeed define a representation of S_t.

Now let $x \in \tilde{N}_p^t$ such that x lies in the l^{th} factor of $\tilde{N}_p^t$. If $l \neq j, j+1$ then
$$T_j \rho(x) T_j^{-1} = \epsilon(x)\rho(x) = \rho(s_j x s_j^{-1}).$$
If $l = j$ then Lemma 7.7 part (3) tells us that $T_j \rho(x) T_j^{-1} = \rho(s_j x s_j^{-1})$ and similarly for $l = j+1$. □

We denote by Exten_t^+ the character of the ρ in the above lemma. We now wish to explicitly describe the character values of Exten_t^+. We denote by $\frac{1}{2}(\tilde{N}_p^t \rtimes S_t)$ the preimage of $(N_p \wr S_t) \cap A_{pt}$ under the natural map $\tilde{N}_p^t \rtimes S_t \twoheadrightarrow N_p \wr S_t$.

We mirror the notation from Section 7.1 for labelling the elements of $\tilde{N}_p^t \rtimes S_t$.

Lemma 7.9 *Let* $0 = n_0 < \cdots < n_m = t$ *and* $x_l \in \tilde{N}_p$ *for* $(1 \le l \le m)$*. Consider*

$$x = \prod_l ((x_l, 1, \dots, 1); (n_{l-1}+1, \dots, n_l)) \in \tilde{N}_p^t \rtimes S_t.$$

If for any l *we have* $x_l \notin \tilde{A}_p$ *then* $\mathrm{Exten}_t^+(x) = 0$*, otherwise*

$$\mathrm{Exten}_t^+(x) = \begin{cases} \prod_{l,(n_l-n_{l-1})odd} \zeta_0(x_l). \prod_{l,(n_l-n_{l-1})even} (\overline{\zeta}_0^+(x_l) - \overline{\zeta}_0^-(x_l)) & \text{if } x \in \frac{1}{2}(\tilde{N}_p^t \rtimes S_t), \\ \prod_l (\overline{\zeta}_0^+(x_l) - \overline{\zeta}_0^-(x_l)) & \text{if } x \notin \frac{1}{2}(\tilde{N}_p^t \rtimes S_t). \end{cases}$$

Proof Let $u_1^+, \dots u_{(p-1)/2}^+$ be a basis of U^+ and $u_1^-, \dots u_{(p-1)/2}^-$ a basis of U^-. Consider the action of x on $U^{\otimes t}$ with respect to the basis

$$\{u_{j_1}^{\epsilon_1} \otimes \cdots \otimes u_{j_t}^{\epsilon_t}\}_{j_l \in [\frac{p-1}{2}], \epsilon_l \in \{\pm 1\}}.$$

We can view the action of x as a Kronecker product of its actions on the vector spaces V_l with bases

$$\{u_{j_{n_{l-1}+1}}^{\epsilon_{n_{l-1}+1}} \otimes \cdots \otimes u_{j_{n_l}}^{\epsilon_{n_l}}\}$$

for varying l. Note that the action of x on V_l is not necessarily the same as the action of $((x_l, 1, \dots, 1); (n_{l-1}+1, \dots, n_l))$ on V_l. $((x_r, 1, \dots, 1); (n_{r-1}+1, \dots, n_r))$ may, for some $r \neq l$, act via $\mathcal{S}^{\otimes(n_l-n_{l-1})}$ on V_l. If this is true for an odd number of $r \neq l$ then "an extra $\mathcal{S}$" acts on V_l. We now write down how x acts on V_l.

$$x.\ (u_{j_{n_{l-1}+1}}^{\epsilon_{n_{l-1}+1}} \otimes \cdots \otimes u_{j_{n_l}}^{\epsilon_{n_l}}) = \eta(x_l.u_{j_{n_l}}^{\epsilon_{n_l}} \otimes u_{j_{n_{l-1}+1}}^{\epsilon_{n_{l-1}+1}} \otimes \cdots \otimes u_{j_{n_l-2}}^{\epsilon_{n_l-2}} \otimes u_{j_{n_l-1}}^{\epsilon_{n_l-1}}),$$

for some $\eta = \pm 1$. First note that if an extra $\mathcal{S}$ acts on V_l then a factor of $\prod_{r=n_{l-1}+1}^{n_l} \epsilon_r$ appears in η. Next we note that

$$(n_{l-1}+1, \dots, n_l) = (n_{l-1}+1, n_{l-1}+2)(n_{l-1}+2, n_{l-1}+3)\dots(n_l-1, n_l).$$

and applying successive T_j's swaps $u_{n_l}^{\epsilon_{n_l}}$ with each of the other $u_{n_j}^{\epsilon_{n_j}}$'s in turn. Therefore if $\epsilon_{n_l} = -1$ then a factor of $\prod_{r=n_{l-1}+1}^{n_l-1} \epsilon_r$ appears due to the repeated action of T. Finally the remaining $\mathcal{S}_j$'s that act for $(n_{l-1}+1 \le j \le n_l)$ give a factor of $\left(\prod_{r=n_{l-1}+1}^{n_l-1} \epsilon_r\right)^{n_l-n_{l-1}-2}$. Putting this all together we get:

$$\eta = \begin{cases} \left(\prod_{r=n_{l-1}+1}^{n_l-1} \epsilon_r\right)^{n_l-n_{l-1}-2} & \text{if } \epsilon_{n_{l-1}+1} = 1 \text{ and no extra } \mathcal{S}, \\ \left(\prod_{r=n_{l-1}+1}^{n_l-1} \epsilon_r\right)^{n_l-n_{l-1}-2}\left(\prod_{r=n_{l-1}+1}^{n_l} \epsilon_r\right) & \text{if } \epsilon_{n_{l-1}+1} = 1 \text{ and extra } \mathcal{S}, \\ \left(\prod_{r=n_{l-1}+1}^{n_l-1} \epsilon_r\right)^{n_l-n_{l-1}-1} & \text{if } \epsilon_{n_{l-1}+1} = -1 \text{ and no extra } \mathcal{S}, \\ \left(\prod_{r=n_{l-1}+1}^{n_l-1} \epsilon_r\right)^{n_l-n_{l-1}-1}\left(\prod_{r=n_{l-1}+1}^{n_l} \epsilon_r\right) & \text{if } \epsilon_{n_{l-1}+1} = -1 \text{ and extra } \mathcal{S}. \end{cases}$$

Now one can see that the only basis vectors v of V_l that could possibly appear with non-zero coefficient in $x.v$ are those of the form $u \otimes \cdots \otimes u$. Since elements of $\tilde{N}_p \backslash (\tilde{N}_p \cap \tilde{A}_p)$ swap U^+ and U^- we have that $\mathrm{Exten}_t^+(x) = 0$ if $x_l \notin \tilde{A}_p$ for some l. So let's assume $x_l \in \tilde{A}_p$ for all l. Now for $j_{n_{l-1}+1} = \cdots = j_{n_l}$ and $\epsilon_{n_{l-1}+1} = \cdots = \epsilon_{n_l}$ we have

$$\eta = \begin{cases} 1 & \text{if } \epsilon_{n_{l-1}+1} = 1 \text{ and no extra } \mathcal{S}, \\ 1 & \text{if } \epsilon_{n_{l-1}+1} = 1 \text{ and extra } \mathcal{S}, \\ (-1)^{n_l-n_{l-1}-1} & \text{if } \epsilon_{n_{l-1}+1} = -1 \text{ and no extra } \mathcal{S}, \\ -1 & \text{if } \epsilon_{n_{l-1}+1} = -1 \text{ and extra } \mathcal{S}. \end{cases}$$

Then $u \otimes \cdots \otimes u$ appears in $x.(u \otimes \cdots \otimes u)$ with coefficient η (as above) multiplied by the coefficient of u in $x_l u$. Therefore, the trace of the action of x on V_l is

$$\begin{cases} \zeta_0(x_l) & \text{if } n_l - n_{l-1} \text{ is odd and no extra } \mathcal{S}, \\ \overline{\zeta}_0^+(x_l) - \overline{\zeta}_0^-(x_l) & \text{otherwise.} \end{cases}$$

The lemma now follows. □

We now wish to show that if $t > 1$ then $\text{Exten}_t^+ \downarrow_{\frac{1}{2}(\tilde{N}_p^t \rtimes S_t)}$ is irreducible. This is equivalent to there existing some $x \in (\tilde{N}_p^t \rtimes S_t) \backslash \frac{1}{2}(\tilde{N}_p^t \rtimes S_t)$ with $\text{Exten}_t^+(x) \neq 0$. Due to Lemma 7.9 it is enough to show that there exists some $x \in \tilde{N}_p \cap \tilde{A}_p$ such that $\overline{\zeta}_0^+(x) \neq \overline{\zeta}_0^-(x)$. This is of course the case. We denote by Exten_t^- the associate of Exten_t^+ with respect to the subgroup $\frac{1}{2}(\tilde{N}_p^t \rtimes S_t) \leq \tilde{N}_p^t \rtimes S_t$.

As a $\mathbb{C}(\tilde{N}_p^t \cap \tilde{A}_{pt})$-module $U^{\otimes t}$ decomposes as

$$\left(\bigoplus_{\prod_j \epsilon_j = 1} \bigotimes_j U_j^{\epsilon_j} \right) \oplus \left(\bigoplus_{\prod_j \epsilon_j = -1} \bigotimes_j U_j^{\epsilon_j} \right).$$

Note that this decomposition respects the action of A_t (or even S_t) and so extends to an irreducible decomposition of $U^{\otimes t}$ as a $\mathbb{C}((\tilde{N}_p \cap \tilde{A}_p) \rtimes A_t)$-module. We label the two characters of these modules $\overline{\text{Exten}_t^+}$ and $\overline{\text{Exten}_t^-}$ respectively. We now describe these two characters.

Lemma 7.10 *Let* $0 = n_0 < \cdots < n_m = t$ *and* $x_l \in \tilde{N}_p$ *for* $(1 \leq l \leq m)$. *Consider*

$$x = \prod_l ((x_l, 1, \ldots, 1); (n_{l-1} + 1, \ldots, n_l)) \in (\tilde{N}_p^t \cap \tilde{A}_{pt}) \rtimes A_t.$$

If for any l *we have* $x_l \notin \tilde{A}_p$ *then* $\overline{\text{Exten}_t^\pm}(x) = 0$, *otherwise*

$$\overline{\text{Exten}_t^+}(x) = \sum_{\prod_{l,(n_l - n_{l-1}) odd} \epsilon_l = 1} \left(\prod_{l,(n_l - n_{l-1}) odd} \overline{\zeta}_0^{\epsilon_l}(x_l) \right) . \prod_{l,(n_l - n_{l-1}) even} (\overline{\zeta}_0^+(x_l) - \overline{\zeta}_0^-(x_l))$$

and

$$\overline{\text{Exten}_t^-}(x) = \sum_{\prod_{l,(n_l - n_{l-1}) odd} \epsilon_l = -1} \left(\prod_{l,(n_l - n_{l-1}) odd} \overline{\zeta}_0^{\epsilon_l}(x_l) \right) . \prod_{l,(n_l - n_{l-1}) even} (\overline{\zeta}_0^+(x_l) - \overline{\zeta}_0^-(x_l)).$$

Proof We proceed as in the proof of Lemma 7.9. For $\epsilon \in \{\pm 1\}$ we define V_l^ϵ to be the subspace of V_l that is the linear span of

$$\{u_{j_{n_{l-1}+1}}^{\epsilon_{n_{l-1}+1}} \otimes \cdots \otimes u_{j_{n_l}}^{\epsilon_{n_l}}\}_{\prod_r \epsilon_r = \epsilon}.$$

Clearly we have $V_l = V_l^+ \oplus V_l^-$ and also

$$\bigoplus_{\prod_j \epsilon_j = 1} \bigotimes_j U_j^{\epsilon_j} = \bigoplus_{\prod_l \epsilon_l = 1} \bigotimes_l V_l^{\epsilon_l} \text{ and } \bigoplus_{\prod_j \epsilon_j = -1} \bigotimes_j U_j^{\epsilon_j} = \bigoplus_{\prod_l \epsilon_l = -1} \bigotimes_l V_l^{\epsilon_l}.$$

Once again we can conclude that $\overline{\mathrm{Exten}_t}^{+}(x) = \overline{\mathrm{Exten}_t}^{-}(x) = 0$ if for some l we have that $x_l \notin \tilde{A}_p$ so let's assume $x_l \in \tilde{A}_p$ for all l. Then the trace of the action of x on V_l^{ϵ} is

$$\begin{cases} \overline{\zeta}_0^{\epsilon}(x_l) & \text{if } n_l - n_{l-1} \text{ is odd,} \\ \overline{\zeta}_0^{+}(x_l) - \overline{\zeta}_0^{-}(x_l) & \text{if } n_l - n_{l-1} \text{ is even and } \epsilon = 1, \\ 0 & \text{if } n_l - n_{l-1} \text{ is even and } \epsilon = -1, \end{cases}$$

and hence the result. □

Now set $M_1 := U^{\otimes t}$ and M_2 the representation space of the character $\xi_\lambda^{(\pm)}$ of $\tilde{S}_t$, where $\lambda \vdash t$. Let $\tilde{N}_p^t$ and $\tilde{S}_t$ act on $M_1 \otimes M_2$ via:

$$\begin{aligned} x : m_1 \otimes m_2 &\mapsto x.m_1 \otimes m_2 \\ s : m_1 \otimes m_2 &\mapsto \theta_t(s).m_1 \otimes s.m_2, \end{aligned}$$

for all $m_1 \in M_1$, $m_2 \in M_2$, $x \in \tilde{N}_p^t$ and $s \in \tilde{S}_t$. As the conjugation action of S_t on $\tilde{N}_p^t$ in $\tilde{N}_p^t \rtimes S_t$ is given, through θ_t, by the action of $\tilde{S}_t$ on $\tilde{N}_p^t$ in $\tilde{N}_p^t\tilde{S}_t$, this defines a representation of $\tilde{N}_p^t\tilde{S}_t$ on $M_1 \otimes M_2$. We denote by $\chi_\lambda^{(\pm)}$ the character of this representation. By Lemma 2.11 $\chi_\lambda^{(\pm)}$ is irreducible and if $x \in \tilde{N}_p^t$ and $s \in \tilde{S}_t$ then $\chi_\lambda^{(\pm)}(xs) = \mathrm{Exten}_t^{+}(x\theta_t(s))\xi_\lambda^{(\pm)}(s)$.

Lemma 7.11 *The irreducible constituents of $\chi_t \uparrow^{\tilde{N}_p^t\tilde{S}_t}$ are precisely the characters*

$$\{\chi_\lambda | \lambda \vdash t, \sigma(\lambda) = 1\} \cup \{\chi_\lambda^{\pm} | \lambda \vdash t, \sigma(\lambda) = -1\}.$$

Furthermore, if $\sigma(\lambda) = 1$ then χ_λ is self-associate and if $\sigma(\lambda) = -1$ then $\chi_\lambda^{\pm}$ are associates.

Proof Set $G := \tilde{N}_p^t\tilde{S}_t$, $H := \tilde{N}_p^t$ and M_1, M_2 to be the representation spaces corresponding to χ_t and $\xi_\lambda^{(\pm)}$ respectively. Lemma 2.11 now says that the required character is irreducible. Next suppose $M = M_1 \otimes M_2$ and $M' = M_1 \otimes M_2'$ are two such representations. If M and M' are isomorphic as $\mathbb{C}G$-modules then they must be isomorphic as $\mathbb{C}H$-modules and hence by Schur's lemma any isomorphism is of the form $1 \otimes \psi$ for some invertible linear map $\psi : M_2 \to M_2'$. Hence, distinct $\xi_\lambda^{(\pm)}$'s give rise to distinct characters $\chi_\lambda^{(\pm)}$. We now show there are no more characters of G appearing. Let e be the idempotent of $\mathbb{C}H$ corresponding to the character χ_t. Then

$$\begin{aligned} \dim(\mathbb{C}Ge) &\geq \sum_{\lambda \vdash t, \sigma(\lambda)=1} \dim(\chi_\lambda)^2 + 2 \sum_{\lambda \vdash t, \sigma(\lambda)=-1} \dim(\chi_\lambda^{+})^2 \\ &= \dim(\chi_t)^2 \left(\sum_{\lambda \vdash t, \sigma(\lambda)=1} \dim(\xi_\lambda)^2 + 2 \sum_{\lambda \vdash t, \sigma(\lambda)=-1} \dim(\xi_\lambda^{+})^2 \right) = \dim(\mathbb{C}He)t! = \dim(\mathbb{C}Ge). \end{aligned}$$

Therefore we have equality throughout and all irreducible constituents of $\chi_t \uparrow^{\tilde{N}_p^t\tilde{S}_t}$ are of the desired form.

For the final part suppose $x \in \tilde{N}_p^t$ and $s \in \tilde{S}_t$. If $\sigma(\lambda) = 1$ then as ξ_λ is self-associate then $\chi_\lambda(xs) = 0$ if $s \in \tilde{S}_t \backslash \tilde{A}_t$. Also if $x \in \tilde{N}_p^t \backslash \tilde{A}_{pt}$ then by Lemma 7.9 $\chi_\lambda(xs) = 0$. If $\sigma(\lambda) = -1$ then as $\xi_\lambda^{\pm}$ is non-self-associate then $\xi_\lambda^{\pm}(g) \neq 0$ for some $g \in \tilde{S}_t \backslash \tilde{A}_t$. Now if in Lemma 7.9 we construct x using $\theta_t(g)$ and all x_l's such that $\overline{\zeta_0}^+(x_l) \neq \overline{\zeta_0}^-(x_l)$ then $\chi_\lambda^\pm(x) \neq 0$ and $x \in \tilde{N}_p^t \tilde{S}_t \backslash \tilde{A}_{pt}$. We now clearly have that $\chi_\lambda^\pm$ are associates. □

Note that if in the definition of $\chi_\lambda^{(\pm)}$ we replace Exten_t^+ by Exten_t^- or we replace $\xi_\lambda^{(\pm)}$ by $\xi_\lambda^{(\mp)}$ then we replace $\chi_\lambda^{(\pm)}$ by $\chi_\lambda^{(\mp)}$.

Let $t > 1$ and $\sigma(\lambda) = 1$. By directly decomposing the representation space of χ_λ we see that we have a character $\overline{\overline{\chi}}_\lambda^{++}$ of $(\tilde{N}_p^t \cap \tilde{A}_{pt})\tilde{A}_t$ given by

$$\overline{\overline{\chi}}_\lambda^{++}(xs) = \overline{\mathrm{Exten}}_t^+(x\theta_t(s))\overline{\xi}_\lambda^+(s) \text{ for all } x \in \tilde{N}_p^t \cap \tilde{A}_{pt} \text{ and } s \in \tilde{A}_t.$$

By comparing dimensions we see that $\overline{\overline{\chi}}_\lambda^{++} \uparrow^{\tilde{N}_p^t \tilde{S}_t} = \chi_\lambda$ and hence $\overline{\overline{\chi}}_\lambda^{++}$ is irreducible. Frobenius reciprocity now gives that $\chi_\lambda \downarrow_{(\tilde{N}_p^t \cap \tilde{A}_{pt})\tilde{A}_t}$ is the sum of four non-isomorphic characters. We label these four characters

$$\begin{aligned} \overline{\overline{\chi}}_\lambda^{++}(xs) &= \overline{\mathrm{Exten}}_t^+(x\theta_t(s))\overline{\xi}_\lambda^+(s) \\ \overline{\overline{\chi}}_\lambda^{+-}(xs) &= \overline{\mathrm{Exten}}_t^+(x\theta_t(s))\overline{\xi}_\lambda^-(s) \\ \overline{\overline{\chi}}_\lambda^{-+}(xs) &= \overline{\mathrm{Exten}}_t^-(x\theta_t(s))\overline{\xi}_\lambda^+(s) \\ \overline{\overline{\chi}}_\lambda^{--}(xs) &= \overline{\mathrm{Exten}}_t^-(x\theta_t(s))\overline{\xi}_\lambda^-(s), \end{aligned}$$

where $x \in \tilde{N}_p^t \cap \tilde{A}_{pt}$ and $s \in \tilde{A}_t$. Note that by the comments preceding 7.10 we have that $\overline{\mathrm{Exten}}_t^+$ is fixed by conujugation by S_t. Therefore conjugating by $s \in \tilde{S}_t \backslash \tilde{A}_t$ takes $\overline{\overline{\chi}}_\lambda^{++}$ to $\overline{\overline{\chi}}_\lambda^{+-}$. Also conjugating by $x \in \tilde{N}_p^t \backslash \tilde{A}_{pt}$ cannot fix $\overline{\overline{\chi}}_\lambda^{++}$ and so must take it to $\overline{\overline{\chi}}_\lambda^{-+}$. We now label the following irreducible characters of $\tilde{N}_p^t \tilde{S}_t \cap \tilde{A}_{pt}$

$$\begin{aligned} \overline{\chi}_\lambda^+ &:= \overline{\overline{\chi}}_\lambda^{++} \uparrow^{(\tilde{N}_p^t \tilde{S}_t)\cap \tilde{A}_{pt}} = \overline{\overline{\chi}}_\lambda^{--} \uparrow^{(\tilde{N}_p^t \tilde{S}_t)\cap \tilde{A}_{pt}} \\ \overline{\chi}_\lambda^- &:= \overline{\overline{\chi}}_\lambda^{+-} \uparrow^{(\tilde{N}_p^t \tilde{S}_t)\cap \tilde{A}_{pt}} = \overline{\overline{\chi}}_\lambda^{-+} \uparrow^{(\tilde{N}_p^t \tilde{S}_t)\cap \tilde{A}_{pt}} . \end{aligned}$$

Now suppose $\sigma(\lambda) = -1$. Again we have characters $\overline{\overline{\chi}}_\lambda^\pm$ of $(\tilde{N}_p^t \cap \tilde{A}_{pt})\tilde{A}_t$ given by

$$\overline{\overline{\chi}}_\lambda^\pm(xs) = \overline{\mathrm{Exten}}_t^\pm(x\theta_t(s))\overline{\xi}_\lambda(s) \text{ for all } x \in \tilde{N}_p^t \cap \tilde{A}_{pt} \text{ and } s \in \tilde{A}_t.$$

By comparing dimensions we see that $\overline{\overline{\chi}}_\lambda^\pm \uparrow^{\tilde{N}_p^t \tilde{S}_t \cap \tilde{A}_{pt}} = \overline{\chi}_\lambda$ and hence $\overline{\overline{\chi}}_\lambda^+$ is irreducible.

We now drop all the assumptions on $\boldsymbol{t} = (t_0, \dots, t_{(p-1)/2})$. Using Lemma 6.2 we will describe all the irreducible constituents of $\chi_{\boldsymbol{t}}^{(\pm)} \uparrow^{\tilde{N}_p^t \tilde{S}_t}$.

Let $\lambda = (\lambda_0, \dots, \lambda_{(p-1)/2})$ be a $(p+1)/2$-tuple of partitions with λ_0 strict and $|\lambda_j| = t_j$ for $(0 \leq j \leq (p-1)/2)$. We set $\boldsymbol{t}(\lambda) := (t_0, \dots, t_{(p-1)/2})$ and define $\sigma(\lambda) := \sigma(\lambda_0)(-1)^{t-t_0}$ and

$$\Delta_t^+ := \{\lambda \in \Delta_t | \sigma(\lambda) = 1\}, \Delta_t^- := \{\lambda \in \Delta_t | \sigma(\lambda) = -1\}.$$

Now using Lemma 7.6 we can construct the character(s) $\chi_{(\varnothing,\lambda_1,\dots,\lambda_{(p-1)/2})}^{(\pm)}$ of $\tilde{N}_p^{t-t_0}\tilde{S}_{(0,t_1,\dots,t_{(p-1)/2})}$. As described in Lemma 7.1 we can and do identify this subgroup with $\tilde{N}_p^{t-t_0}\tilde{S}_{(0,t_1,\dots,t_{(p-1)/2})}[t_0]$. Using Lemma 7.11 we also have the character(s) $\chi_{\lambda_0}^{(\pm)}$ of $\tilde{N}_p^{t_0}\tilde{S}_{t_0}$. With the labelling convention of Lemma 6.2 we can construct the character(s) $\chi_\lambda^{(\pm)}$ of $\tilde{N}_p^t \tilde{S}_{\boldsymbol{t}}$. Now due to Theorem 2.9 and Remark 2.10 we have that $\chi_\lambda^{(\pm)} \uparrow^{\tilde{N}_p^t \tilde{S}_t}$ is irreducible and we denote this character by $\chi^{\lambda(\pm)}$.

Theorem 7.12 *A complete list of irreducible spin characters of* $\tilde{N}_p^t\tilde{S}_t$ *is given by*

$$\{\chi^\lambda|\lambda \in \Delta_t^+\} \cup \{\chi^{\lambda\pm}|\lambda \in \Delta_t^-\}.$$

Proof If $\boldsymbol{t} = (t_0, \ldots, t_{(p-1)/2})$ then due to Theorem 2.9, Remark 2.10 and Lemma 7.4 all we have to show is that

$$\{\chi_\lambda|\lambda \in \Delta_t^+, \boldsymbol{t}(\lambda) = \boldsymbol{t}\} \cup \{\chi_\lambda^\pm|\lambda \in \Delta_t^-, \boldsymbol{t}(\lambda) = \boldsymbol{t}\}$$

is a complete list of irreducible constituents of $\chi_{\boldsymbol{t}}^{(+)} \uparrow^{\tilde{N}_p^t\tilde{S}_t}$ if $t_0 > 1$ or of $\chi_{\boldsymbol{t}}^{(\pm)} \uparrow^{\tilde{N}_p^t\tilde{S}_t}$ if $t_0 \leq 1$. Let's set $G := \tilde{N}_p^t\tilde{S}_t$ and $H := \tilde{N}_p^t$. First suppose $t_0 > 1$ and $t - t_0$ is even and let e be the idempotent of $\mathbb{C}H$ corresponding to the character $\chi_{\boldsymbol{t}}$. Then by Lemmas 6.2, 7.6 and 7.11

$$\begin{aligned}
&\dim(\mathbb{C}Ge) \geq \sum_{\lambda\in\Delta_t^+,\boldsymbol{t}(\lambda)=\boldsymbol{t}} \dim(\chi_\lambda)^2 + \sum_{\lambda\in\Delta_t^-,\boldsymbol{t}(\lambda)=\boldsymbol{t}} (\dim(\chi_\lambda^+)^2 + \dim(\chi_\lambda^-)^2)\\
&= \sum_{\lambda_0\in\mathcal{D}_t^+,\chi\in\mathrm{Irr}(S_{t_1}\times\cdots\times S_{t_{(p-1)/2}})} (\dim(\xi_\lambda)\dim(\chi)\dim(\chi_{\boldsymbol{t}}))^2 +\\
&\quad \sum_{\lambda_0\in\mathcal{D}_t^-,\chi\in\mathrm{Irr}(S_{t_1}\times\cdots\times S_{t_{(p-1)/2}})} \big((\dim(\xi_\lambda^+)\dim(\chi)\dim(\chi_{\boldsymbol{t}}))^2 + (\dim(\xi_\lambda^-)\dim(\chi)\dim(\chi_{\boldsymbol{t}}))^2\big)\\
&= t_0!t_1!\ldots t_{(p-1)/2}!\dim(\chi_{\boldsymbol{t}})^2 = t_0!t_1!\ldots t_{(p-1)/2}!\dim(\mathbb{C}He) = \dim(\mathbb{C}Ge).
\end{aligned}$$

Therefore we have equality throughout and all irreducible constituents of $\chi_{\boldsymbol{t}} \uparrow^{\tilde{N}_p^t\tilde{S}_t}$ are of the desired form. Now let $t_0 > 1$, $t - t_0$ be odd and e_+ and e_- the idempotents of $\mathbb{C}H$ corresponding to the characters $\chi_{\boldsymbol{t}}^+$ and $\chi_{\boldsymbol{t}}^-$ respectively. Then similarly to above

$$\begin{aligned}
&\dim(\mathbb{C}G(e_+ + e_-)) \geq \sum_{\lambda\in\Delta_t^+,\boldsymbol{t}(\lambda)=\boldsymbol{t}} \dim(\chi_\lambda)^2 + \sum_{\lambda\in\Delta_t^-,\boldsymbol{t}(\lambda)=\boldsymbol{t}} (\dim(\chi_\lambda^+)^2 + \dim(\chi_\lambda^-)^2)\\
&= \sum_{\lambda_0\in\mathcal{D}_t^-,\chi\in\mathrm{Irr}(S_{t_1}\times\cdots\times S_{t_{(p-1)/2}})} (2\dim(\xi_\lambda^+)\dim(\chi)\dim(\chi_{\boldsymbol{t}}^+))^2 +\\
&\quad \sum_{\lambda_0\in\mathcal{D}_t^+,\chi\in\mathrm{Irr}(S_{t_1}\times\cdots\times S_{t_{(p-1)/2}})} \big((\dim(\xi_\lambda)\dim(\chi)\dim(\chi_{\boldsymbol{t}}^+))^2 + (\dim(\xi_\lambda)\dim(\chi)\dim(\chi_{\boldsymbol{t}}^-))^2\big)\\
&= 2t_0!t_1!\ldots t_{(p-1)/2}!\dim(\chi_{\boldsymbol{t}}^+)^2 = 2t_0!t_1!\ldots t_{(p-1)/2}!\dim(\mathbb{C}He_+) = \dim(\mathbb{C}G(e_+ + e_-)).
\end{aligned}$$

Therefore we have equality throughout and all irreducible constituents of $\chi_{\boldsymbol{t}}^+ \uparrow^{\tilde{N}_p^t\tilde{S}_t}$ are of the desired form. Similar calculations prove the result for $t_0 \leq 1$. □

8 Results on Character Values

In this section we prove many results about the character values of $\tilde{N}_p^t\tilde{S}_t$.

Lemma 8.1 *Let $a \in \tilde{N}_p \cap \tilde{A}_p$ be an element of order p. Then*

$$\overline{\zeta}_0^+(a) = \frac{-1 \pm i^{\frac{p-1}{2}}\sqrt{p}}{2}.$$

Proof By the comments at the beginning of Section 7.2 there exists a faithful linear spin character ψ of $\theta^{-1}(C_p)$ with $\overline{\zeta}_0^+ = \psi \uparrow^{\tilde{N}_p \cap \tilde{A}_p}$. Then $\psi(a) = \omega_p$ for some primitive p^{th} root of unity ω_p. We set

$$c := \overline{\zeta}_0^+(a) = \sum_{j=1}^{(p-1)/2} \omega_p^{b^{2j}},$$

where b is a generator of the multiplicative group of $\mathbb{F}_p^\times$. We define the field automorphism

$$\begin{aligned} \tau : \mathbb{Q}(w_p) &\to \mathbb{Q}(w_p) \\ w_p &\mapsto w_p^b. \end{aligned}$$

Then

$$c + \tau(c) = -1 \text{ and } (c - \tau(c))^2 = \left(\frac{-1}{p}\right)p = (-1)^{\frac{p-1}{2}} p,$$

as $(c - \tau(c))$ is a quadratic Gauss sum. This gives

$$c = \frac{-1 \pm i^{\frac{p-1}{2}}\sqrt{p}}{2}.$$

□

Lemma 8.2 *Let $a \in \tilde{N}_p$ have p'-order. Then $C_{\tilde{N}_p}(a) \nsubseteq \tilde{A}_p$. In particular $\overline{\zeta}_0^+(a) = \overline{\zeta}_0^-(a)$.*

Proof Recall the definition of y_1 from Section 7.1. Suppose $b \in \tilde{N}_p$ with $\theta_p(b) = y_1$. All p'-elements of N_p are conjugate to a power of y_1 and so all p'-elements of $\tilde{N}_p$ are conjugate to a power of b or z times a power of b and hence are centralised by a conjugate of b. The first part is now clear. For the second part let $g \in C_{\tilde{N}_p}(a) \backslash \tilde{A}_p$. Then $\overline{\zeta}_0^-(a) = \overline{\zeta}_0^+(gag^{-1}) = \overline{\zeta}_0^+(a)$. □

Recall the definition of $\chi_n^{C\pm}$ from Lemma 5.1.

Lemma 8.3 *Let q be an odd positive integer and $\tau = o((1, \dots, q)) \in \tilde{S}_q$. Then*

$$\chi_q^{C\pm}(\phi_q(\tau)) = (-1)^{(q^2-1)/8}.$$

Proof By Lemma 5.2

$$\phi_q(\tau) = \pm \frac{1}{2^{\frac{q-1}{2}}}(e_1 + e_2) \dots (e_{q-1} + e_q).$$

It is not clear which sign is correct but since $(1, \ldots, q)^2$ is conjugate to $(1, \ldots, q)$ in S_q, τ^2 is conjugate to τ in $\tilde{S}_q$ and so we calculate

$$\chi_q^{\mathcal{C}\pm}\bigg(\bigg(\frac{1}{2^{\frac{q-1}{2}}}(e_1+e_2)\ldots(e_{q-1}+e_q)\bigg)^2\bigg).$$

We now proceed by induction on q. The result is clearly true for $q = 1$. By the presentation of $S_n^{\pm}$ in §4 and by Lemma 5.2 for $q = 3$ we have

$$\chi_3^{\mathcal{C}\pm}(\phi_3(\tau)) = \chi_3^{\mathcal{C}\pm}(-\frac{1}{2}(e_1+e_2)(e_2+e_3)) = \frac{1}{2}.-2 = -1 = (-1)^{\frac{3^2-1}{8}}.$$

Now we note that since $\phi_q(\tau) \in \mathcal{C}_q^+$,

$$\begin{aligned}\chi_q^{\mathcal{C}\pm}(\phi_q(\tau)) &= 2^{\frac{q-1}{2}}\chi_\varnothing\bigg(\bigg(\frac{1}{2^{\frac{q-1}{2}}}(e_1+e_2)\ldots(e_{q-1}+e_q)\bigg)^2\bigg)\\ &= \frac{1}{2^{\frac{q-1}{2}}}\chi_\varnothing(((e_1+e_2)\ldots(e_{q-1}+e_q))^2),\end{aligned}$$

where $\chi_\varnothing(x)$ is equal to the coefficient of $e_\varnothing$ in x. Now

$$\begin{aligned}&\chi_\varnothing([(e_1+e_2)\ldots(e_{q-1}+e_q)][(e_1+e_2)\ldots(e_{q-1}+e_q)])\\ =\ &\chi_\varnothing((e_1+e_2)(e_2+e_3)(e_1+e_2)[(e_3+e_4)\ldots(e_{q-1}+e_q)][(e_2+e_3)\ldots(e_{q-1}+e_q)])\\ =\ &\chi_\varnothing((2e_1-2e_3)[(e_3+e_4)\ldots(e_{q-1}+e_q)][(e_2+e_3)\ldots(e_{q-1}+e_q)])\\ =\ &2\chi_\varnothing((-e_3)[(e_3+e_4)\ldots(e_{q-1}+e_q)][e_3(e_3+e_4)\ldots(e_{q-1}+e_q)])\\ =\ &2\chi_\varnothing((-e_3)(e_3+e_4)(-e_3)[(e_4+e_5)\ldots(e_{q-1}+e_q)][(e_3+e_4)\ldots(e_{q-1}+e_q)])\\ =\ &2\chi_\varnothing((e_3-e_4)[(e_4+e_5)\ldots(e_{q-1}+e_q)][(e_3+e_4)\ldots(e_{q-1}+e_q)])\\ =\ &2\chi_\varnothing((e_3-e_4)(e_4+e_5)(e_3+e_4)[(e_5+e_6)\ldots(e_{q-1}+e_q)][(e_4+e_5)\ldots(e_{q-1}+e_q)])\\ =\ &2\chi_\varnothing((-2e_4-2e_3e_4e_5)[(e_5+e_6)\ldots(e_{q-1}+e_q)][(e_4+e_5)\ldots(e_{q-1}+e_q)])\\ =\ &-4\chi_\varnothing(e_4[(e_5+e_6)\ldots(e_{q-1}+e_q)]e_4[(e_5+e_6)\ldots(e_{q-1}+e_q)])\\ =\ &-4\chi_\varnothing([(e_5+e_6)\ldots(e_{q-1}+e_q)][(e_5+e_6)\ldots(e_{q-1}+e_q)])\end{aligned}$$

and the claim follows by induction. □

Lemma 8.4 *Let $x \in \tilde{N}_p^t\tilde{S}_t$ be of type $(\pi_0, \ldots, \pi_{(p-1)/2})$ with π_j having at least one even part for some even j. Suppose further that $x \in \tilde{N}_p^t\tilde{S}_t \cap \tilde{A}_{pt}$ or π_j does not have distinct parts. Then x is conjugate to zx in $\tilde{N}_p^t\tilde{S}_t$.*

Proof If $x \in \tilde{A}_{pt}$ then we can assume without loss of generality that $x = \prod_{j=1}^l x_j$ is a disjoint cycle decomposition of x with $x_1 = ((a, 1, \ldots, 1); \tau)$, where τ is in the preimage, under θ_t, of a cycle of length m for some positive even integer m and $a \in \tilde{N}_p \cap \tilde{A}_p$. In particular $x_1 \notin \tilde{A}_{pt}$ and so $\prod_{j=2}^l x_j \notin \tilde{A}_{pt}$. Therefore by [16, p.172], viewing x_1 and $\prod_{j=2}^l x_j$ as elements of $\tilde{S}_{pt}$, we have $x_1\big(\prod_{j=2}^l x_j\big) = z\big(\prod_{j=2}^l x_j\big)x_1$ and hence $x_1^{-1}xx_1 = zx$.

If π_j does not have distinct parts then we may assume by the first paragraph that $x \notin \tilde{A}_{pt}$ and that $x = \prod_{j=1}^l x_j$ is a disjoint cycle decomposition of x where $x_1 = ((a, 1, \ldots, 1); \tau_1)$ and $x_2 = ((a, 1, \ldots, 1); \tau_2)$, where $a \in \tilde{N}_p \cap \tilde{A}_p$, $\theta_t(\tau_1) = (1, \ldots, m)$ and $\theta_t(\tau_2) =$

$(m+1,\dots,2m)$ for some positive integer m. In particular $x_1x_2\tilde{A}_{pt}$ and hence $\prod_{j=3}^{l} x_j \notin \tilde{A}_{pt}$. Now let $u \in \theta_t^{-1}((1, m+1))\dots(m, 2m))$. Surely $\theta_t(u)\theta_t(x_1)\theta_t(u)^{-1} = \theta_t(x_2)$ and so $ux_1u^{-1} = x_2$ or zx_2 and $ux_1x_2u^{-1} = x_2x_1$. If m is even then $x_2x_1 = zx_1x_2$ and $u(\prod_{j=3}^{l} x_j)u^{-1} = \prod_{j=3}^{l} x_j$. If m is odd then $x_2x_1 = x_1x_2$ and $u(\prod_{j=3}^{l} x_j)u^{-1} = z\prod_{j=3}^{l} x_j$. Either way $uxu^{-1} = zx$. $\square$

Now for the rest of the section let $\lambda \in \Delta_t$ and set $\boldsymbol{t}(\lambda) =: (t_0, \dots, t_{(p-1)/2})$. Now let

$$x_0 \in \tilde{N}_p^{t_0}, \quad s_0 \in \tilde{S}_{t_0}, x' \in \tilde{N}_p^{t-t_0}[t_0], s'^{\in \tilde{S}}_{(0,t_1,\dots,t_{(p-1)/2})}[t_0]$$

and set $x := x_0s_0x's' \in \tilde{N}_p^t\tilde{S}_{\boldsymbol{t}(\lambda)}$. We will identify $\tilde{N}_p^{t-t_0}[t_0]$ with $\tilde{N}_p^{t-t_0}$ and $\tilde{S}_{(0,t_1,\dots,t_{(p-1)/2})}[t_0]$ with $\tilde{S}_{(0,t_1,\dots,t_{(p-1)/2})}$ via Lemma 7.1.

Lemma 8.5 *If $\lambda \in \Delta_t^-$, $t_0 > 0$ and x is p-regular then $\chi_\lambda^+(x) = \chi_\lambda^-(x)$.*

Proof First note that by Lemma 7.2 if x is of type π then $\pi_0 = \varnothing$. If $\sigma(\lambda_0) = -1$ and $t - t_0$ is even then by Lemma 7.9 $\chi_\lambda^+(x) \neq \chi_\lambda^-(x)$ only if $x_0s_0 \notin \tilde{A}_{pt_0}$ and $x's' \in \tilde{A}_{p(t-t_0)}$. By Lemmas 7.9 and 8.2 we note that $\chi_{\lambda_0}^{\pm}(x_0) \neq 0$ only if $x_0 \in \tilde{A}_{pt_0}$ and s_0 has only cycles of odd length but this is a contradiction as $x_0s_0 \notin \tilde{A}_{pt_0}$.

Next suppose $\sigma(\lambda) = 1 and t - t_0$ is odd. Similarly to above we can assume $x_0 \in \tilde{A}_{pt}$, $s_0 \in \tilde{A}_{pt}$ and $x's' \notin \tilde{A}_{p(t-t_0)}$. By Lemmas 7.10 and 8.2 we deduce that $\overline{\overline{\chi}}_{\lambda_0}^{++}(x_0s_0) = \overline{\overline{\chi}}_{\lambda_0}^{-+}(x_0s_0)$ and $\overline{\overline{\chi}}_{\lambda_0}^{+-}(x_0s_0) = \overline{\overline{\chi}}_{\lambda_0}^{--}(x_0s_0)$ and therefore $\overline{\chi}_{\lambda_0}^{+}(x_0s_0) = \overline{\chi}_{\lambda_0}^{-}(x_0s_0)$ and so $\chi_\lambda^+(x) = \chi_\lambda^-(x)$. $\square$

Recall the definition of the function f from Section 7.1.

Lemma 8.6 *Suppose $t_0 = 0$ and x has disjoint cycle decomposition $x = \prod_{j=1}^{l} x_j$, with $f(\theta_{pt}(x_1)) \in N_p \cap A_p$.*

1. *If $\lambda \in \Delta_t^-$ then $\chi_\lambda^+(x) = \chi_\lambda^-(x)$.*
2. *If $\lambda \in \Delta_t^+$ and $x \in \tilde{A}_{pt}$ then $\overline{\chi}_\lambda^+(x) = \overline{\chi}_\lambda^-(x)$.*

Proof Let $\phi_{\boldsymbol{t}(\lambda)} : \tilde{N}_p^t\tilde{S}_{\boldsymbol{t}(\lambda)} \to \mathcal{C}_t$ be the homomorphism in Lemma 7.5. Then the coefficient of $e_{[t]}$ in $\phi_{\boldsymbol{t}(\lambda)}(x)$ is zero. The result now follows from 5.1. $\square$

Lemma 8.7 *Let $\lambda \in \Delta_t^-$ and $x \notin \tilde{A}_{pw}$ with $x's'$ having disjoint cycle decomposition $x's' = \prod_{j=1}^{l} x_j$. Then $\chi_\lambda^{\pm}(x) = 0$ unless x_0s_0 has type $(\lambda_0, \varnothing, \dots, \varnothing)$ and $f(\theta_{pt}(x_j)) \in N_p \backslash A_p$ for $(1 \le j \le l)$ in which case $\chi_\lambda^{\pm}(x) \in \sqrt{p^{l(\lambda_0)}}\mathcal{R}$.*

Proof We denote by $\lambda\backslash\{0\} \in \Delta_{t-t_0}$ the multipartition obtained from λ by deleting λ_0. We first note that if $f(\theta_{pt}(x_j)) \in N_p \cap A_p$ for some j then by Lemma 8.6 $\chi_{\lambda\backslash\{0\}}^+(x's') = \chi_{\lambda\backslash\{0\}}^-(x's')$ if $\lambda\backslash\{0\} \in \Delta_{t-t_0}^-$ or $\overline{\chi}_{\lambda\backslash\{0\}}^+(x's') = \overline{\chi}_{\lambda\backslash\{0\}}^-(x's')$ if $\lambda\backslash\{0\} \in \Delta_{t-t_0}^+$ and $x's' \in \tilde{A}_{p(t-t_0)}$. So, by Lemma 6.2, we have that $\chi_\lambda^+(x) = \chi_\lambda^-(x)$ and hence as $x \notin \tilde{A}_{pw}$, $\chi_\lambda^+(x) = \chi_\lambda^-(x) = 0$.

Now suppose $\sigma(\lambda_0) = 1$ and $t - t_0$ is odd. By Lemmas 6.2 and 7.10 we can assume that $x_0 \in \tilde{N}_p^{t_0} \cap \tilde{A}_{pt_0}$, $s_0 \in \tilde{A}_{t_0}$ and $x's' \notin \tilde{A}_{p(t-t_0)}$. Now

$$\chi_\lambda^{\pm}(x) = \pm(\overline{\mathrm{Exten}}_{t_0}^{+}(x_0\theta_t(s_0)) - \overline{\mathrm{Exten}}_{t_0}^{-}(x_0\theta_t(s_0)))(\overline{\xi}_{\lambda_0}^{+}(s_0) - \overline{\xi}_{\lambda_0}^{-}(s_0))\chi_{\lambda\backslash\{0\}}^{+}(x's').$$

Therefore by Theorem 4.4 $\chi_\lambda^{\pm}(x) \neq 0$ only if s_0 is of type λ_0. Also by Lemma 7.10

$$\overline{\mathrm{Exten}}_{t_0}^{+}(x_0\theta_t(s_0)) - \overline{\mathrm{Exten}}_{t_0}^{-}(x_0\theta_t(s_0)) \neq 0$$

only if x_0s_0 is of type $(\lambda_0, \varnothing, \dots, \varnothing)$ and in this case by Lemma 8.1 we have that $\overline{\mathrm{Exten}}_{t_0}^{+}(x_0\theta_t(s_0)) - \overline{\mathrm{Exten}}_{t_0}^{-}(x_0\theta_t(s_0)) \in \sqrt{p^{l(\lambda_0)}}\mathcal{R}$.

Next suppose $\sigma(\lambda_0) = -1$ and $t - t_0$ is even. By Lemma 6.2 we can assume $x_0s_0 \notin \tilde{A}_{pt_0}$ and $x's' \in \tilde{A}_{p(t-t_0)}$. Now

$$\chi_\lambda^{\pm}(x) = \pm\,\mathrm{Exten}_{t_0}^{+}(x_0\theta_t(s_0))\xi_\lambda(s_0)(\overline{\chi}_{\lambda\backslash\{0\}}^{+}(x's') - \overline{\chi}_{\lambda\backslash\{0\}}^{-}(x's')).$$

So by Lemma 7.9 $\chi_\lambda^{\pm}(x) \neq 0$ only if $x_0 \in \tilde{A}_{pt_0}$. In this case $s_0 \notin \tilde{A}_{t_0}$ and hence by Theorem 4.4 we can assume s_0 is of type λ_0. Therefore as in the previous paragraph x_0s_0 is of type $(\lambda_0, \varnothing, \dots, \varnothing)$ and $\chi_\lambda^{\pm}(x) \in \sqrt{p^{l(\lambda_0)}}\mathcal{R}$. □

Lemma 8.8 *Let $\lambda \in \Delta_t^+$ and $x \in \tilde{A}_{pw}$ with $x's'$ having disjoint cycle decomposition $x's' = \prod_{j=1}^{l} x_j$. Then $\overline{\chi}_\lambda^{+}(x) = \overline{\chi}_\lambda^{-}(x)$ unless x_0s_0 has type $(\lambda_0, \varnothing, \dots, \varnothing)$ and $f(\theta_{pt}(x_j)) \in N_p\backslash A_p$ for $(1 \leq j \leq l)$ in which case $\overline{\chi}_\lambda^{+}(x) - \overline{\chi}_\lambda^{-}(x) \in \sqrt{p^{l(\lambda_0)}}\mathcal{R}$.*

Proof We first note that if $f(\theta_{pt}(x_j)) \in N_p \cap A_p$ for some j then by Lemmas 6.2 and 8.6 we have that $\overline{\chi}_\lambda^{+}(x) = \overline{\chi}_\lambda^{-}(x)$.

Now suppose $\sigma(\lambda_0) = 1$ and $t - t_0$ is even. By Lemmas 6.3 and 7.10 we can assume $x_0 \in \tilde{A}_{pt_0}$ and $s_0 \in \tilde{A}_{t_0}$ and so $x's' \in \tilde{A}_{p(t-t_0)}$. Now

$$\overline{\chi}_\lambda^{+}(x) - \overline{\chi}_\lambda^{-}(x) = (\overline{\chi}_{\lambda_0}^{+}(x_0s_0) - \overline{\chi}_{\lambda_0}^{-}(x_0s_0))(\overline{\chi}_{\lambda\backslash\{0\}}^{+}(x's') - \overline{\chi}_{\lambda\backslash\{0\}}^{-}(x's'))$$

and

$$(\overline{\chi}_{\lambda_0}^{+}(x_0s_0) - \overline{\chi}_{\lambda_0}^{-}(x_0s_0)) = (\overline{\mathrm{Exten}}_{t_0}^{+}(x_0\theta_t(s_0)) - \overline{\mathrm{Exten}}_{t_0}^{-}(x_0\theta_t(s_0)))(\overline{\xi}^{+}(s_0) - \overline{\xi}^{-}(s_0)).$$

Therefore by Theorem 4.4 $\overline{\chi}_\lambda^{+}(x) - \overline{\chi}_\lambda^{-}(x) \neq 0$ only if s_0 is of type λ_0 and the result then follows by Lemma 7.10.

Now suppose $\sigma(\lambda_0) = -1$ and $t - t_0$ is odd. By Lemmas 6.3 and 7.10 we can assume that $x_0\tilde{A}_{pt_0}$ and $s_0 \notin \tilde{A}_{t_0}$ and so $x's' \notin \tilde{A}_{p(t-t_0)}$. Then

$$\overline{\chi}_\lambda^{+}(x) - \overline{\chi}_\lambda^{-}(x) = 2i\chi_{\lambda_0}^{+}(x_0s_0)\chi_{\lambda\backslash\{0\}}^{+}(x's')$$

Therefore by Theorem 4.4 $\overline{\chi}_\lambda^{+}(x) - \overline{\chi}_\lambda^{-}(x) \neq 0$ only if s_0 is of type λ_0 and the result then follows by Lemma 7.10. □

Lemma 8.9 *Let $\lambda \in \Delta_t^+$ with $t_0 > 0$ and let $x \in \tilde{N}_p^t\tilde{S}_{t(\lambda)} \cap \tilde{A}_{pt}$ be p-regular. Then $\overline{\chi}_\lambda^{+}(x) = \overline{\chi}_\lambda^{-}(x)$.*

Proof Let $x = x_0x'$ with $x_0 \in \tilde{N}_p^{t_0}\tilde{S}_{t_0}$ and $x' \in \tilde{N}_p^{t-t_0}(\tilde{S}_{t_1} \dots \tilde{S}_{t_{(p-1)/2}})$. Now let $x_0 = \prod_{j=1}^{l} x_j$ be a disjoint cycle decomposition. Then by Lemma 7.9 $f(\theta_{pt}(x_j))$ is p-regular for all j.

First suppose $\sigma(\lambda_0) = 1$ and $t - t_0$ is even. Then by Lemma 6.3 we may assume $x_0 \in \tilde{A}_{pt_0}$ and $x' \in \tilde{A}_{p(t-t_0)}$. By Lemmas 7.10 and 8.2 $\overline{\chi}^+_{\lambda_0}(x_0) = \overline{\chi}^-_{\lambda_0}(x_0)$ and therefore that $\overline{\chi}^+_\lambda(x) = \overline{\chi}^-_\lambda(x)$.

Next suppose $\sigma(\lambda_0) = -1$ and $t - t_0$ is odd. By Lemma 6.3 the result is clear unless $x_0 \notin \tilde{A}_{pt_0}$ and $x' \notin \tilde{A}_{p(t-t_0)}$. However, again by Lemma 7.10 in this case $\chi^+_{\lambda_0}(x_0) = 0$ and hence $\overline{\chi}^+_\lambda(x) = \overline{\chi}^-_\lambda(x) = 0$. □

9 Murnaghan-Nakayama Rules

By Murnaghan-Nakayama rule we mean the maps r^{xs} in part (4) of Definition 2.2. For the original Murnaghan-Nakayama rule for the symmetric group (see [8, Theorem 2.4.7]). In this section we present the corresponding rule for the double covers of the symmetric and alternating groups as proved by M. Cabanes and O. Brunat and J. Gramain respectively. We then go on to prove that $\tilde{N}^t_p\tilde{S}_t$ also possesses a Murnaghan-Nakayama rule.

9.1 Murnaghan-Nakayama Rule for $\tilde{S}_n$ *and* $\tilde{A}_n$

Theorem 9.1 [3, Thoerem 20] *Let n and q be positive integers with q odd and $q \leq n$. We write $\tau = o((1, \ldots, q)) \in \tilde{S}_n$. Let $x \in \tilde{S}_{n-q}[q]$ which we identify with $\tilde{S}_{n-q}$ via Lemma 7.1. Then*

$$\xi^{(\pm)}_\lambda(\tau x) = \sum_{\mu \in M_{\overline{q}}(\lambda), \sigma(\mu)=1} a(\xi^{(\pm)}_\lambda, \xi_\mu)\xi_\mu(x) + \sum_{\mu \in M_{\overline{q}}(\lambda), \sigma(\mu)=-1} a(\xi^{(\pm)}_\lambda, \xi^+_\mu)\xi^+_\mu(x) + a(\xi^{(\pm)}_\lambda, \xi^-_\mu)\xi^-_\mu(x),$$

where

$$a(\xi^{(\varepsilon)}_\lambda, \xi^{(\eta)}_\mu) := \begin{cases} (-1)^{\frac{q^2-1}{8}}\alpha^\lambda_\mu & \text{if } \sigma(\mu) = 1, \\ \frac{1}{2}(-1)^{\frac{q^2-1}{8}}\alpha^\lambda_\mu & \text{if } \sigma(\lambda) = 1, \sigma(\mu) = -1, \\ \frac{1}{2}(-1)^{\frac{q^2-1}{8}}(\alpha^\lambda_\mu + \varepsilon\eta i^{\frac{q-1}{2}}\sqrt{q}) & \text{if } \sigma(\lambda) = \sigma(\mu) = -1, \end{cases}$$

and

$$\alpha^\lambda_\mu = (-1)^{L(b)}2^{m(b)},$$

where $L(b)$ is the leg length of the q-bar b removed from λ to get μ, and

$$m(b) := \begin{cases} 1 & \text{if } \sigma(\lambda) = 1, \sigma(\mu) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

There is also a natural labelling of the characters such that the following theorem for $\tilde{A}_n$ holds. We also adopt this labelling for the remainder of this paper.

Theorem 9.2 [2, Theorem 4.14] *Let τ and x be as above with $x \in \tilde{A}_{n-q}$. Then*

$$\overline{\xi}_\lambda^{(\pm)}(\tau x) = \sum_{\mu \in M_{\overline{q}}(\lambda), \sigma(\mu)=-1} a(\overline{\xi}_\lambda^{(\pm)}, \overline{\xi}_\mu)\overline{\xi}_\mu(x) + \sum_{\mu \in M_{\overline{q}}(\lambda), \sigma(\mu)=1} a(\overline{\xi}_\lambda^{(\pm)}, \overline{\xi}_\mu^{+})\overline{\xi}_\mu^{+}(x) + a(\overline{\xi}_\lambda^{(\pm)}, \overline{\xi}_\mu^{-})\overline{\xi}_\mu^{-}(x).$$

where

$$a(\overline{\xi}_\lambda^{(\varepsilon)}, \overline{\xi}_\mu^{(\eta)}) := \begin{cases} (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda & \textit{if } \sigma(\lambda) = -1, \\ \frac{1}{2}(-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda & \textit{if } \sigma(\lambda) = 1, \sigma(\mu) = -1, \\ \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda + \varepsilon\eta i^{\frac{q-1}{2}} \sqrt{q}) & \textit{if } \sigma(\lambda) = \sigma(\mu) = 1, \end{cases}$$

and the α_μ^λ are as in Theorem 9.1.

9.2 Murnaghan-Nakayama Rule for $\tilde{N}_p^t \tilde{S}_t$

Let t be a positive integer and $\lambda \in \Delta_t$ with $\boldsymbol{t}(\lambda) = (t_0, \ldots, t_{(p-1)/2})$. We set

$$\lambda\backslash\{j\} = (\lambda_0, \ldots, \lambda_{j-1}, \varnothing, \lambda_{j+1}, \ldots, \lambda_{(p-1)/2}).$$

If μ is any partition then we set

$$(\mu, \lambda\backslash\{j\}) = (\lambda_0, \ldots, \lambda_{j-1}, \mu, \lambda_{j+1}, \ldots, \lambda_{(p-1)/2}).$$

Theorem 9.3 *Let q be an odd positive integer with $q \leq t$ and set $\tau = o((1, \ldots, q))$. Let $a \in \tilde{N}_p \cap \tilde{A}_p$ be such that*

$$\overline{\zeta}_0^{+}(a) = \frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2},$$

and set $g := (\boldsymbol{a}; \tau) \in \tilde{N}_p^t \tilde{S}_t$, where $\boldsymbol{a} = (a, 1, \ldots, 1)$. Let $x \in \tilde{N}_p^{t-q} \tilde{S}_{t-q}[q]$ which we identify with $\tilde{N}_p^{t-q} \tilde{S}_{t-q}$ as in Lemma 7.1. Now suppose $gx \in \tilde{N}_p^t \tilde{S}_{\boldsymbol{t}(\lambda)}$.

1. *If $\sigma(\lambda_0) = 1$ and $t - t_0$ is even then*

$$\chi_\lambda(gx) = -\Bigg(\sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=1}} a(\xi_{\lambda_0}, \xi_\mu)\chi_{(\mu,\lambda\backslash\{0\})}(x) + \sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=-1}} a(\xi_{\lambda_0}, \xi_\mu^{\pm})(\chi^{+}_{(\mu,\lambda\backslash\{0\})}(x) + \chi^{-}_{(\mu,\lambda\backslash\{0\})}(x))\Bigg).$$

2. *If $\sigma(\lambda_0) = -1$ and $t - t_0$ is even then*

$$\chi_\lambda^{+}(gx) = -\sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=1}} a(\xi_{\lambda_0}^{\pm}, \xi_\mu)\chi_{(\mu,\lambda\backslash\{0\})}(x) +$$

$$\sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=-1}} \Bigg[\Bigg(a(\xi_{\lambda_0}^{+}, \xi_\mu^{+})\Bigg(\frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}\Bigg) + a(\xi_{\lambda_0}^{+}, \xi_\mu^{-})\Bigg(\frac{-1 - i^{\frac{p-1}{2}}\sqrt{p}}{2}\Bigg)\Bigg)\chi^{+}_{(\mu,\lambda\backslash\{0\})}(x)$$

$$+\Bigg(a(\xi_{\lambda_0}^{+}, \xi_\mu^{+})\Bigg(\frac{-1 - i^{\frac{p-1}{2}}\sqrt{p}}{2}\Bigg) + a(\xi_{\lambda_0}^{+}, \xi_\mu^{-})\Bigg(\frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}\Bigg)\Bigg)\chi^{-}_{(\mu,\lambda\backslash\{0\})}(x)\Bigg].$$

3. *If $\sigma(\lambda_0) = 1$ and $t - t_0$ is odd then*

$$\chi_\lambda^+(gx) = -\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\overline{\xi}^{\pm}_{\lambda_0}, \overline{\xi}_\mu)\chi_{(\mu,\lambda\setminus\{0\})}(x) +$$

$$\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} \Bigg[\left(a(\overline{\xi}^{+}_{\lambda_0}, \overline{\xi}^{+}_\mu)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\overline{\xi}^{+}_{\lambda_0}, \overline{\xi}^{-}_\mu)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi^+_{(\mu,\lambda\setminus\{0\})}(x)$$

$$+\left(a(\overline{\xi}^{+}_{\lambda_0}, \overline{\xi}^{+}_\mu)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\overline{\xi}^{+}_{\lambda_0}, \overline{\xi}^{-}_\mu)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi^-_{(\mu,\lambda\setminus\{0\})}(x)\Bigg].$$

4. *If $\sigma(\lambda_0) = -1$ and $t - t_0$ is odd then*

$$\chi_\lambda(gx) = -\Bigg(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0}, \xi_\mu)(\chi^+_{(\mu,\lambda\setminus\{0\})}(x) + \chi^-_{(\mu,\lambda\setminus\{0\})}(x))$$

$$+ \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} (a(\xi^{+}_{\lambda_0}, \xi^{+}_\mu) + a(\xi^{+}_{\lambda_0}, \xi^{-}_\mu))\chi_{(\mu,\lambda\setminus\{0\})}(x)\Bigg).$$

Proof Let

$$x_0 \in \tilde{N}_p^{t_0-q}[q], \qquad s_0 \in \tilde{S}_{t_0-q}[q], \qquad x'^{\in \tilde{N}_p^{t-t_0}[t_0]}, \qquad s'^{\in \tilde{S}_{(0,t_1,\dots,t_{(p-1)/2})}[t_0]},$$

with $x = x_0 s_0 x' s'$. As always we identify all the above subgroups with their non-shifted counterpart via Lemma 7.1.

1. By Lemma 6.2

$$\chi_\lambda(gx) = \begin{cases} \chi_{\lambda_0}(gx_0s_0)\overline{\chi}^+_{\lambda\setminus\{0\}}(x's') + \chi_{\lambda_0}(gx_0s_0)\overline{\chi}^-_{\lambda\setminus\{0\}}(x's') & \text{if } x's' \in \tilde{A}_{p(t-t_0)},\\ 0 & \text{otherwise.}\end{cases}$$

Now by Lemma 7.9 and Theorem 9.1

$$\chi_{\lambda_0}(gx_0s_0) = \xi_{\lambda_0}(\tau s_0)\,\mathrm{Exten}^+_{t_0}(\boldsymbol{a}\theta_{t_0}(\tau)x_0\theta_{t_0}(s_0))$$

$$= \Bigg(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0}, \xi_\mu)\xi_\mu(s_0) + \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\xi_{\lambda_0}, \xi^{\pm}_\mu)(\xi^+_\mu(s_0) + \xi^-_\mu(s_0))\Bigg)$$

$$\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2} + \epsilon(s_0)\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\mathrm{Exten}^+_{t_0-q}(x_0\theta_{t_0-q}(s_0))$$

$$= -\Bigg(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0}, \xi_\mu)\chi_\mu(x_0s_0) + \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\xi_{\lambda_0}, \xi^{\pm}_\mu)(\chi^+_\mu(x_0s_0) + \chi^-_\mu(x_0s_0))\Bigg).$$

Therefore, by Lemma 6.2, we have that

$$\chi_\lambda(gx) = -\Bigg(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0},\xi_\mu)\chi_{(\mu,\lambda\setminus\{0\})}(x)$$
$$\sum_{\substack{+\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\xi_{\lambda_0},\xi_\mu^{\pm})(\chi^+_{(\mu,\lambda\setminus\{0\})}(x)+\chi^-_{(\mu,\lambda\setminus\{0\})}(x))\Bigg).$$

2. By Lemma 6.2

$$\chi_\lambda^+(gx) = \begin{cases} \chi^+_{\lambda_0}(gx_0s_0)\overline{\chi}^+_{\lambda\setminus\{0\}}(x's') + \chi^-_{\lambda_0}(gx_0s_0)\overline{\chi}^-_{\lambda\setminus\{0\}}(x's') & \text{if } x's'\in\tilde{A}_{p(t-t_0)},\\ 0 & \text{otherwise.}\end{cases}$$

Now by Lemma 7.9 and Theorem 9.1

$$\chi^+_{\lambda_0}(gx_0s_0) = \xi^+_{\lambda_0}(\tau s_0)\,\mathrm{Exten}^+_{t_0}(\boldsymbol{a}\theta_{t_0}(\tau)x_0\theta_{t_0}(s_0))$$
$$= \Bigg(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0},\xi_\mu)\xi_\mu(s_0) + \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} (a(\xi^+_{\lambda_0},\xi^+_\mu)\xi^+_\mu(s_0)+a(\xi^+_{\lambda_0},\xi^-_\mu)\xi^-_\mu(s_0))\Bigg)$$
$$\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}+\epsilon(s_0)\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\mathrm{Exten}^+_{t_0-q}(x_0\theta_{t_0-q}(s_0))$$
$$= -\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0},\xi_\mu)\chi_\mu(x_0s_0) +$$
$$\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} \Bigg[\Bigg(a(\xi^+_{\lambda_0},\xi^+_\mu)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\xi^+_{\lambda_0},\xi^-_\mu)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)\chi^+_\mu(x_0s_0) +$$
$$\Bigg(a(\xi^+_{\lambda_0},\xi^+_\mu)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\xi^+_{\lambda_0},\xi^-_\mu)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)\chi^-_\mu(x_0s_0)\Bigg].$$

Therefore if $x's'\in\tilde{A}_{p(t-t_0)}$ we have that

$$\chi^+_\lambda(gx) = -\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0},\xi_\mu)\chi_\mu(x_0s_0)(\overline{\chi}^+_{\lambda\setminus\{0\}}(x's')+\overline{\chi}^-_{\lambda\setminus\{0\}}(x's')) +$$
$$\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} \Bigg[\Bigg(a(\xi^+_{\lambda_0},\xi^+_\mu)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\xi^+_{\lambda_0},\xi^-_\mu)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)$$
$$(\chi^+_\mu(x_0s_0)\overline{\chi}^+_{\lambda\setminus\{0\}}(x's')+\chi^-_\mu(x_0s_0)\overline{\chi}^-_{\lambda\setminus\{0\}}(x's')) +$$
$$\Bigg(a(\xi^+_{\lambda_0},\xi^+_\mu)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\xi^+_{\lambda_0},\xi^-_\mu)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)$$
$$(\chi^+_\mu(x_0s_0)\overline{\chi}^-_{\lambda\setminus\{0\}}(x's')+\chi^-_\mu(x_0s_0)\overline{\chi}^+_{\lambda\setminus\{0\}}(x's'))\Bigg],$$

and so by Lemma 6.2

$$\chi_\lambda^+(gx) = -\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0}^{\pm},\xi_\mu)\chi_{(\mu,\lambda\backslash\{0\})}(x) + \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} \Bigg[\left(a(\xi_{\lambda_0}^+,\xi_\mu^+)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi_{\lambda_0}^+,\xi_\mu^-)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi_{(\mu,\lambda\backslash\{0\})}^+(x) + \left(a(\xi_{\lambda_0}^+,\xi_\mu^+)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi_{\lambda_0}^+,\xi_\mu^-)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi_{(\mu,\lambda\backslash\{0\})}^-(x)\Bigg].$$

3. By Lemma 6.2

$$\chi_\lambda^+(gx) = \begin{cases} \overline{\chi}_{\lambda_0}^+(gx_0s_0)\chi_{\lambda\backslash\{0\}}^+(x's') + \overline{\chi}_{\lambda_0}^-(gx_0s_0)\chi_{\lambda\backslash\{0\}}^-(x's') & \text{if } x_0s_0 \in \tilde{A}_{p(t_0-q)},\\ 0 & \text{otherwise,}\end{cases}$$

and if $x_0s_0 \in \tilde{A}_{p(t_0-q)}$ then by Lemma 6.3

$$\overline{\chi}_{\lambda_0}^+(gx_0s_0) = \begin{cases} \overline{\xi}_{\lambda_0}^+(\tau s_0)\overline{\mathrm{Exten}}_{t_0}^+(\boldsymbol{a}\theta_{t_0}(\tau)x_0\theta_{t_0}(s_0)) + & \text{if } x_0 \in \tilde{A}_{p(t_0-q)} \text{ and } s_0 \in \tilde{A}_{t_0-q},\\ \overline{\xi}_{\lambda_0}^-(\tau s_0)\overline{\mathrm{Exten}}_{t_0}^-(\boldsymbol{a}\theta_{t_0}(\tau)x_0\theta_{t_0}(s_0)) & \\ 0 & \text{otherwise.}\end{cases}$$

Then by Lemma 7.10 and Theorem 9.2 if $x_0 \in \tilde{A}_{p(t_0-q)}$ and $s_0 \in \tilde{A}_{t_0-q}$

$$\begin{aligned}
&\overline{\xi}_{\lambda_0}^+(\tau s_0)\overline{\mathrm{Exten}}_{t_0}^+(\boldsymbol{a}\theta_{t_0}(\tau)x_0\theta_{t_0}(s_0))\\
&= \left(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\overline{\xi}_{\lambda_0}^+,\overline{\xi}_\mu^+)\overline{\xi}_\mu^+(s_0) + a(\overline{\xi}_{\lambda_0}^+,\overline{\xi}_\mu^-)\overline{\xi}_\mu^-(s_0) + \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\overline{\xi}_{\lambda_0}^{\pm},\overline{\xi}_\mu)\overline{\xi}_\mu(s_0)\right)\\
&\left[\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\overline{\mathrm{Exten}}_{t_0-q}^+(x_0\theta_{t_0-q}(s_0)) + \left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\overline{\mathrm{Exten}}_{t_0-q}^-(x_0\theta_{t_0-q}(s_0))\right]\\
&= \left(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\overline{\xi}_{\lambda_0}^+,\overline{\xi}_\mu^+)\overline{\overline{\chi}}_\mu^{++}(x_0s_0) + a(\overline{\xi}_{\lambda_0}^+,\overline{\xi}_\mu^-)\overline{\overline{\chi}}_\mu^{+-}(x_0s_0) + \right.\\
&\left.\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\overline{\xi}_{\lambda_0}^{\pm},\overline{\xi}_\mu)\overline{\overline{\chi}}_\mu^+(x_0s_0)\right)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) +\\
&\left(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\overline{\xi}_{\lambda_0}^+,\overline{\xi}_\mu^+)\overline{\overline{\chi}}_\mu^{-+}(x_0s_0) + a(\overline{\xi}_{\lambda_0}^+,\overline{\xi}_\mu^-)\overline{\overline{\chi}}_\mu^{--}(x_0s_0) + \right.\\
&\left.\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\overline{\xi}_{\lambda_0}^{\pm},\overline{\xi}_\mu)\overline{\overline{\chi}}_\mu^-(x_0s_0)\right)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right).
\end{aligned}$$

Therefore,

$$\chi_\lambda^+(gx) = \sum_{\substack{\eta_1,\eta_2,\eta_3,\eta_4,\eta_5\in\{\pm1\}\\ \eta_1\eta_2\eta_3\eta_4\eta_5=1}} \Bigg[\Bigg(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\overline{\xi}_{\lambda_0}^+, \overline{\xi}_\mu^{\eta_1})\overline{\overline{\chi}}_\mu^{\eta_2\eta_3}(x_0s_0)\chi_{\lambda\setminus\{0\}}^{\eta_4}(x's') + \frac{1}{2}\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\overline{\xi}_{\lambda_0}^\pm, \overline{\xi}_\mu)\overline{\overline{\chi}}_\mu^{\eta_2}(x_0s_0)\chi_{\lambda\setminus\{0\}}^{\eta_4}(x's')\Bigg)\left(\frac{-1+\eta_5 i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg]$$

if $x_0 \in \tilde{A}_{p(t_0-q)}$ and $s_0 \in \tilde{A}_{t_0-q}$ and 0 otherwise. Therefore by Lemma **??**

$$\begin{aligned}\chi_\lambda^+(gx) = &- \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\overline{\xi}_{\lambda_0}^\pm, \overline{\xi}_\mu)\chi_{(\mu,\lambda\setminus\{0\})}(x) + \\ &\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} \Bigg[\Bigg(\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)a(\overline{\xi}_{\lambda_0}^+, \overline{\xi}_\mu^+) + \left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)a(\overline{\xi}_{\lambda_0}^+, \overline{\xi}_\mu^-)\Bigg)\chi_{(\mu,\lambda\setminus\{0\})}^+(x) \\ &+\Bigg(\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)a(\overline{\xi}_{\lambda_0}^+, \overline{\xi}_\mu^+) + \left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)a(\overline{\xi}_{\lambda_0}^+, \overline{\xi}_\mu^-)\Bigg)\chi_{(\mu,\lambda\setminus\{0\})}^+(x)\Bigg].\end{aligned}$$

4. By Lemma 6.2

$$\chi_\lambda(gx) = \begin{cases} \chi_{\lambda_0}^+(gx_0s_0)\overline{\chi}_{\lambda\setminus\{0\}}(x's') + \chi_{\lambda_0}^-(gx_0s_0)\overline{\chi}_{\lambda\setminus\{0\}}(x's') & \text{if } x's' \in \tilde{A}_{t-t_0},\\ 0 & \text{otherwise.}\end{cases}$$

Now by Lemma 7.9 and Theorem 9.1

$$\begin{aligned}\chi_{\lambda_0}^+(gx_0s_0) &= \xi_{\lambda_0}^+(\tau s_0)\,\mathrm{Exten}_{t_0}^+(\boldsymbol{a}\theta_{t_0}(\tau)x_0\theta_{t_0}(s_0)) \\ &= \Bigg(\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0}^\pm, \xi_\mu)\xi_\mu(s_0) + \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} (a(\xi_{\lambda_0}^+, \xi_\mu^+)\xi_\mu^+(s_0) + a(\xi_{\lambda_0}^+, \xi_\mu^-)\xi_\mu^-(s_0))\Bigg) \\ &\quad\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2} + \epsilon(s_0)\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\mathrm{Exten}_{t_0-q}^+(x_0\theta_{t_0-q}(s_0)) \\ &= -\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0}^\pm, \xi_\mu)\chi_\mu(x_0s_0) + \\ &\quad\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} \Bigg[\Bigg(a(\xi_{\lambda_0}^+, \xi_\mu^+)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi_{\lambda_0}^+, \xi_\mu^-)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)\chi_\mu^+(x_0s_0) + \\ &\quad\Bigg(a(\xi_{\lambda_0}^+, \xi_\mu^+)\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi_{\lambda_0}^+, \xi_\mu^-)\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)\chi_\mu^-(x_0s_0))\Bigg].\end{aligned}$$

Therefore if $x's' \in \tilde{A}_{t-t_0}$

$$\chi_\lambda(gx) = -2 \sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0}, \xi_\mu)\chi_\mu(x_0 s_0)\overline{\chi}_{\lambda\setminus\{0\}}(x's')$$

$$+ \sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=-1}} \Bigg[\Bigg(a(\xi^+_{\lambda_0}, \xi^+_\mu)\left(\frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi^+_{\lambda_0}, \xi^-_\mu)\left(\frac{-1 - i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)$$

$$(\chi^+_\mu(x_0 s_0) + \chi^-_\mu(x_0 s_0)) +$$

$$\Bigg(a(\xi^+_{\lambda_0}, \xi^+_\mu)\left(\frac{-1 - i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi^+_{\lambda_0}, \xi^-_\mu)\left(\frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)$$

$$(\chi^+_\mu(x_0 s_0) + \chi^-_\mu(x_0 s_0))\Bigg]\overline{\chi}_{\lambda\setminus\{0\}}(x's')$$

and so by Lemma 6.2 we have that

$$\begin{aligned}
\chi_\lambda(gx) = &- \sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0}, \xi_\mu)(\chi^+_{(\mu,\lambda\setminus\{0\})}(x) + \chi^-_{(\mu,\lambda\setminus\{0\})}(x)) + \\
&\sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=-1}} \Bigg[\Bigg(a(\xi^+_{\lambda_0}, \xi^+_\mu)\left(\frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi^+_{\lambda_0}, \xi^-_\mu)\left(\frac{-1 - i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg) \\
&+\Bigg(a(\xi^+_{\lambda_0}, \xi^+_\mu)\left(\frac{-1 - i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\xi^+_{\lambda_0}, \xi^-_\mu)\left(\frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\Bigg)\Bigg]\chi_{(\mu,\lambda\setminus\{0\})}(x) \\
= &- \Bigg(\sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0}, \xi_\mu)(\chi^+_{(\mu,\lambda\setminus\{0\})}(x) + \chi^-_{(\mu,\lambda\setminus\{0\})}(x)) + \\
&\sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=-1}} (a(\xi^+_{\lambda_0}, \xi^+_\mu) + a(\xi^+_{\lambda_0}, \xi^-_\mu))\chi_{(\mu,\lambda\setminus\{0\})}(x)\Bigg).
\end{aligned}$$

□

Theorem 9.4 *Let q be an odd positive integer, $(1 \le l \le (p-1)/2)$ and set $t'_l = \sum_{j=0}^{l-1} t_j$. Now let $g := (\boldsymbol{a}; \tau) \in \tilde{N}^t_p \tilde{S}_t$ with $\boldsymbol{a} = (a, 1, \ldots, 1)$, where*

$$\overline{\zeta}^+_0(a) = \frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}$$

and $\tau = [t'_l](o((1, \ldots, q)))$. Let

$$x \in (\tilde{N}^{t'_l}_p \tilde{S}_{t'_l}).(\tilde{N}^{t-t'_l-q}_p \tilde{S}_{t-t'_l-q}[t'_l + q]).$$

We identify this subgroup with $(\tilde{N}^{t'_l}_p \tilde{S}_{t'_l}).(\tilde{N}^{t-t'_l-q}_p \tilde{S}_{t-t'_l-q}[t'_l])$ by identifying $\tilde{N}^{t-t'_l-q}_p \tilde{S}_{t-t'_l-q}[t'_l + q]$ with $\tilde{N}^{t-t'_l-q}_p \tilde{S}_{t-t'_l-q}[t'_l]$ through $\tilde{N}^{t-t'_l-q}_p \tilde{S}_{t-t'_l-q}$ via Lemma 7.1. Now suppose $gx \in \tilde{N}^t_p \tilde{S}_{\boldsymbol{t}}$.

1. *If* $\lambda \in \Delta_t^+$ *then*

$$\chi_\lambda(gx) = (-1)^{\frac{q^2-1}{8}} \sum_{\mu \in M_q(\lambda_l)} (-1)^{L(c_\mu^{\lambda_l})} (\chi^+_{(\mu,\lambda\setminus\{l\})}(x) + \chi^-_{(\mu,\lambda\setminus\{l\})}(x)).$$

2. *If* $\lambda \in \Delta_t^-$ *then*

$$\chi^+_\lambda(gx) = (-1)^{\frac{q^2-1}{8}} \sum_{\mu \in M_q(\lambda_l)} (-1)^{L(c_\mu^{\lambda_l})} \chi_{(\mu,\lambda\setminus\{l\})}(x).$$

Proof Let

$$x_0 \in \tilde{N}_p^{t_0-q}[q], \qquad s_0 \in \tilde{S}_{t_0-q}[q], \qquad x' \in \tilde{N}_p^{t-t_0}[t_0], \qquad s' \in \tilde{S}_{(0,t_1,\dots,t_{(p-1)/2})}[t_0],$$

with $x = x_0 s_0 x' s'$. As always we identify all the above subgroups with their non-shifted counterpart via Lemma 7.1. First let $t - t_0$ be even. Then

$$\chi_{\lambda\setminus\{0\}}(gx's') = \chi_{\overline{\boldsymbol{t}(\lambda\setminus\{0\})}}(\boldsymbol{a}\tau x's')\,\mathrm{Sym}_{\lambda\setminus\{0\}}(\theta_{t-t_0}(\tau)\theta_{t-t_0}(s')),$$

which by the construction of $\chi_{\overline{\boldsymbol{t}(\lambda\setminus\{0\})}}$, Lemma 8.3 and [15, Theorem 4.4] this is equal to

$$\begin{aligned}
&= (-1)^{\frac{q^2-1}{8}} \sum_{\mu \in M_q(\lambda_l)} \Big(\big(\chi^+_{\overline{\boldsymbol{t}((\mu,\lambda\setminus\{l\})\setminus\{0\})}}(x's') + \chi^-_{\overline{\boldsymbol{t}((\mu,\lambda\setminus\{l\})\setminus\{0\})}}(x's')\big) \\
&\quad (-1)^{L(c_\mu^{\lambda_l})}\,\mathrm{Sym}_{((\mu,\lambda\setminus\{l\})\setminus\{0\})}(\theta_{t-t_0-q}(s')) \Big) \\
&= (-1)^{\frac{q^2-1}{8}} \sum_{\mu \in M_q(\lambda_l)} (-1)^{L(c_\mu^{\lambda_l})} \big(\chi^+_{((\mu,\lambda\setminus\{l\})\setminus\{0\})}(x's') + \chi^-_{((\mu,\lambda\setminus\{l\})\setminus\{0\})}(x's')\big).
\end{aligned}$$

Similarly if $t - t_0$ is odd then

$$\chi^\pm_{\lambda\setminus\{0\}}(gx's') = (-1)^{\frac{q^2-1}{8}} \sum_{\mu \in M_q(\lambda_l)} (-1)^{L(c_\mu^{\lambda_l})} \chi_{((\mu,\lambda\setminus\{l\})\setminus\{0\})}(x's').$$

Also if $x's' \in \tilde{A}_{p(t-t_0-q)}$ we have that

$$\overline{\chi}^\pm_{\lambda\setminus\{0\}}(gx's') = (-1)^{\frac{q^2-1}{8}} \sum_{\mu \in M_q(\lambda_l)} (-1)^{L(c_\mu^{\lambda_l})} \overline{\chi}_{((\mu,\lambda\setminus\{l\})\setminus\{0\})}(x's').$$

if $t - t_0$ is even and

$$\overline{\chi}_{\lambda\setminus\{0\}}(gx's') = (-1)^{\frac{q^2-1}{8}} \sum_{\mu \in M_q(\lambda_l)} (-1)^{L(c_\mu^{\lambda_l})} (\overline{\chi}^+_{((\mu,\lambda\setminus\{l\})\setminus\{0\})}(x's') + \overline{\chi}^-_{((\mu,\lambda\setminus\{l\})\setminus\{0\})}(x's'))$$

if $t - t_0$ is odd. The theorem now follows from Lemma 6.2 and applying the relevant above equation. □

We now have a Murnaghan-Nakayama rule for $\tilde{N}_p^t \tilde{S}_t$.

Theorem 9.5 *Let q be an odd positive integer. Set $\tau = o((1,\dots,q))$ and $g := (\boldsymbol{a};\tau) \in \tilde{N}_p^t \tilde{S}_t$, where $\boldsymbol{a} = (a, 1, \dots, 1)$ and*

$$\overline{\zeta}_0^+(a) = \frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2}.$$

Let $x \in \tilde{N}_p^{t-q}\tilde{S}_{t-q}[q]$ *which we identify with* $\tilde{N}_p^{t-q}\tilde{S}_{t-q}$ *via Lemma 7.1.*

1. *If* $\sigma(\lambda_0) = 1$ *and* $t - t_0$ *is even then*

$$\begin{aligned}\chi^{\lambda}(gx) \;=\; -\Bigg(& \sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0},\xi_\mu)\chi^{(\mu,\lambda\backslash\{0\})}(x)\\ &+\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\xi_{\lambda_0},\xi_\mu^{\pm})(\chi^{(\mu,\lambda\backslash\{0\})+}(x)+\chi^{(\mu,\lambda\backslash\{0\})-}(x))\Bigg)\\ &+(-1)^{\frac{q^2-1}{8}}\Bigg[\sum_{j=1}^{\frac{p-1}{2}}\sum_{\mu\in M_q(\lambda_j)}(-1)^{L(c_\mu^{\lambda_j})}(\chi^{(\mu,\lambda\backslash\{j\})+}(x)+\chi^{(\mu,\lambda\backslash\{j\})-}(x))\Bigg].\end{aligned}$$

2. *If* $\sigma(\lambda_0) = -1$ *and* $t - t_0$ *is even then*

$$\begin{aligned}\chi^{\lambda+}(gx) = &-\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}} a(\xi_{\lambda_0}^{\pm},\xi_\mu)\chi^{(\mu,\lambda\backslash\{0\})}(x)\\ +\textstyle\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}}\Bigg[&\left(a(\xi_{\lambda_0}^{+},\xi_\mu^{+})\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\xi_{\lambda_0}^{+},\xi_\mu^{-})\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi^{(\mu,\lambda\backslash\{0\})+}(x)\\ &+\left(a(\xi_{\lambda_0}^{+},\xi_\mu^{+})\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\xi_{\lambda_0}^{+},\xi_\mu^{-})\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi^{(\mu,\lambda\backslash\{0\})-}(x)\Bigg]\\ &+(-1)^{\frac{q^2-1}{8}}\Bigg[\sum_{j=1}^{\frac{p-1}{2}}\sum_{\mu\in M_q(\lambda_j)}(-1)^{L(c_\mu^{\lambda_j})}\chi^{(\mu,\lambda\backslash\{j\})}(x)\Bigg].\end{aligned}$$

3. *If* $\sigma(\lambda_0) = 1$ *and* $t - t_0$ *is odd then*

$$\begin{aligned}\chi^{\lambda+}(gx) = &-\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=-1}} a(\overline{\xi}_{\lambda_0}^{\pm},\overline{\xi}_\mu)\chi^{(\mu,\lambda\backslash\{0\})}(x)+\\ \textstyle\sum_{\substack{\mu\in M_{\overline{q}}(\lambda_0)\\ \sigma(\mu)=1}}\Bigg[&\left(a(\overline{\xi}_{\lambda_0}^{+},\overline{\xi}_\mu^{+})\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\overline{\xi}_{\lambda_0}^{+},\overline{\xi}_\mu^{-})\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi^{(\mu,\lambda\backslash\{0\})+}(x)\\ &+\left(a(\overline{\xi}_{\lambda_0}^{+},\overline{\xi}_\mu^{+})\left(\frac{-1-i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)+a(\overline{\xi}_{\lambda_0}^{+},\overline{\xi}_\mu^{-})\left(\frac{-1+i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\right)\chi^{(\mu,\lambda\backslash\{0\})-}(x)\Bigg]\\ &+(-1)^{\frac{q^2-1}{8}}\Bigg[\sum_{j=1}^{\frac{p-1}{2}}\sum_{\mu\in M_q(\lambda_j)}(-1)^{L(c_\mu^{\lambda_j})}\chi^{(\mu,\lambda\backslash\{j\})}(x)\Bigg].\end{aligned}$$

4. *If* $\sigma(\lambda_0) = -1$ *and* $t - t_0$ *is odd then*

$$\chi^{\lambda}(gx) = -\Bigg(\sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=1}} a(\xi^{\pm}_{\lambda_0}, \xi_\mu)(\chi^{(\mu,\lambda\backslash\{0\})+}(x) + \chi^{(\mu,\lambda\backslash\{0\})-}(x))$$
$$+ \sum_{\substack{\mu \in M_{\overline{q}}(\lambda_0) \\ \sigma(\mu)=-1}} (a(\xi^{+}_{\lambda_0}, \xi^{+}_\mu) + a(\xi^{+}_{\lambda_0}, \xi^{-}_\mu))\chi^{(\mu,\lambda\backslash\{0\})}(x)\Bigg)$$
$$+(-1)^{\frac{q^2-1}{8}}\Bigg[\sum_{j=1}^{\frac{p-1}{2}} \sum_{\mu \in M_q(\lambda_j)} (-1)^{L(c^{\lambda_j}_\mu)}(\chi^{(\mu,\lambda\backslash\{j\})+}(x) + \chi^{(\mu,\lambda\backslash\{j\})-}(x))\Bigg].$$

Proof Apply the proof [15, Theorem 4.4] using Theorems 9.3 and 9.4. □

10 The Broué Perfect Isometry

In this section we construct a Broué perfect isometry between a block of $\tilde{S}_n$ with abelian defect and weight w and positive sign and the block of spin characters of $\tilde{N}^w_p \tilde{S}_w$. We also construct a Broué perfect isometry between the corresponding blocks of $\tilde{A}_n$ and $\tilde{N}^w_p \tilde{S}_w \cap \tilde{A}_{pw}$. We will then use these isometries to construct the Broué perfect isometry of Theorem 1.1.

Let n be a positive integer. We denote by C the set of elements of $\tilde{S}_n$ in conjugacy classes labelled by partitions of n with no part an odd multiple of p. We define S_{S_n} to be the set of elements of S_n all of whose cycles have length 1 or an odd multiple of p and $S := \{o(g) | g \in S_{S_n}\}$.

Let C' be the set of conjugacy classes of $\tilde{N}^w_p \tilde{S}_w$ labelled by $(\pi_0, \dots, \pi_{p-1})$ with π_0 having only even parts. We define $S'_{N_p \wr S_w}$ to be the set of elements of $N_p \wr S_w$ in conjugacy classes labelled by $(\pi_0, \varnothing, \dots, \varnothing, \pi_{p-1})$, where π_0 has only odd parts and $\pi_{p-1} = (1, \dots, 1)$. Note that if $g \in S'_{N_p \wr S_w}$ then g has odd order and so $o(g)$ makes sense if we view $\tilde{N}^w_p \tilde{S}_w$ as a subgroup of $\tilde{S}_{pw}$. We now set $S' := \{o(g) | g \in S'_{N_p \wr S_w}\}$.

Now suppose $\beta = (\beta_1, \beta_2, \dots)$ has only odd parts and $p|\beta| \leq n$. Then β labels a conjugacy class of S. We pick a representative s_β for this conjugacy class to be

$$o((1, \dots, p\beta_1)(p\beta_1 + 1, \dots, p(\beta_1 + \beta_2)) \dots).$$

If in addition $|\beta| \leq w$ then β labels a conjugacy class of S'. We pick a representative s'_β for this conjugacy class to be

$$\prod_l ((a, 1, \dots, 1); \tau_l),$$

where

$$\overline{\zeta}^{+}_0(a) = \frac{-1 + i^{\frac{p-1}{2}}\sqrt{p}}{2} \qquad \text{and} \qquad \tau_l = o\left(\left(\left(\sum_{j=1}^{l-1}\beta_j\right) + 1, \dots, \sum_{j=1}^{l}\beta_j\right)\right) \text{ for all } l.$$

Let $\gamma \in \mathcal{D}^+_{n-p|w|}$ for some $w \in \mathbb{N}_0$ be a p-bar core. In particular γ labels a p-block of $\tilde{S}_n$ of weight w. Let E_γ denote the set of strict partitions (not necessarily of n) with p-bar core γ. We define the bijection

$$\begin{aligned}\Psi : E_\gamma &\to \sqcup_t \Delta_t\\ \lambda &\mapsto \lambda^{(\overline{p})}.\end{aligned}$$

Note that adding any p-bar hook to a partition λ changes the sign of $\sigma(\lambda)$ unless the hook is a part equal to p. Then as $\sigma(\gamma) = 1$ by induction we see that $\sigma(\lambda) = \sigma(\Psi(\lambda))$.

Theorem 10.1 *When $w < p$ then I is a Broué perfect isometry, where I is defined as follows:*

$$\begin{aligned} I : \mathbb{Z}\,\mathrm{Irr}(\tilde{S}_{n,\gamma}) &\to \mathbb{Z}\,\mathrm{IrrSp}(\tilde{N}_p^w \tilde{S}_w)\\ \xi_\lambda &\mapsto \delta_{\overline{p}}(\lambda)(-1)^{w\frac{p^2-1}{8}+w+|\Psi(\lambda)_0|}\chi^{\Psi(\lambda)} && \textit{if } \sigma(\lambda) = 1,\\ \xi_\lambda^\epsilon &\mapsto \delta_{\overline{p}}(\lambda)(-1)^{w\frac{p^2-1}{8}+w+|\Psi(\lambda)_0|}\chi^{\Psi(\lambda)\eta_{\overline{p}}(\lambda)} && \textit{if } \sigma(\lambda) = -1,\end{aligned}$$

where,

$$\eta_{\overline{p}}(\lambda) = \begin{cases} \epsilon\delta_{\overline{p}}(\lambda)(-1)^{w+\frac{p-1}{4}(|\Psi(\lambda)_0|-l(\Psi(\lambda)_0))} & \textit{if } \sigma(\Psi(\lambda)_0) = 1,\\ \epsilon\delta_{\overline{p}}(\lambda)(-1)^{w+\frac{p-1}{4}(|\Psi(\lambda)_0|-l(\Psi(\lambda)_0)+1)} & \textit{if } \sigma(\Psi(\lambda)_0) = -1.\end{cases}$$

Proof We set $G := \tilde{S}_n$ and $G' := \tilde{N}_p^w \tilde{S}_w$. First we note that by [2, Lemma 4.16] $\tilde{S}_{n,\gamma}$ is a union of C-blocks of G. Next we note that if χ (respectively ψ) is an irreducible non-spin (respectively spin) character of G' then $\langle \mathrm{res}_{C'}(\chi), \mathrm{res}_{C'}(\psi)\rangle_{G'} = 0$ and so $\mathrm{IrrSp}(\tilde{N}_p^w \tilde{S}_w)$ is a union of C'-blocks of G.

By [2, Proposition 4.17] G has an MN-structure with respect to C and $\tilde{S}_{n,\gamma}$. Now let $x \in G'$ have disjoint cycle decomposition $\prod_l (x_l; \tau_l)$. We partition the product $x = \prod_{l\in L_0} x_l . \prod_{l\in L_1} x_l$ with $l \in L_0$ if and only if $f(\theta_{pw}(x_l))$ is conjugate to y_0 (see Section 7.1) in N_p. Then, by multiplying both products by z if necessary, we can assume $\prod_{l\in L_0} x_l \in S'$ and $\prod_{l\in L_1} x_l \in C'$. We set

$$G'_{s'_\beta} := \tilde{N}_p^{w-|\beta|}\tilde{S}_{w-|\beta|}[|\beta|].$$

Then by iteratively applying Theorem 9.5 we have an MN-structure for G' with respect to C' and $\mathrm{IrrSp}(\tilde{N}_p^w \tilde{S}_w)$. Therefore part (1) of Theorem 2.3 holds for G and G'.

By an abuse of notation we allow I to denote the map in the statement of the theorem for varying w. (Note that we are including $w = 0$. In this case we define $\tilde{N}_p^0 \tilde{S}_0$ to be the group of order 2 generated by z so that the only irreducible spin character is the non-trivial linear character that is labelled by $(\varnothing, \dots, \varnothing)$.) Let β be a partition having only odd parts and $p|\beta| \leq n$. We first note that if $|\beta| > w$ then by induction and Theorem 9.1 $r^\beta = 0$. Let's now assume $|\beta| \leq w$. We wish to show that $I \circ r^\beta = r'^\beta \circ I$. By induction we can assume $l(\beta) = 1$. Let $\beta = (q)$ for some odd positive integer q and $\lambda \vdash n$.

First suppose $\sigma(\Psi(\lambda)_0) = 1$ and $w - |\Psi(\lambda)_0|$ is even. If $\mu \in M_{\overline{pq}}(\lambda)$ with $\sigma(\mu) = 1$ then μ must be gotten from λ by removing a part of λ equal to pq. By Theorem 9.5 ξ_μ appears in $r^{\beta}(\xi_{\lambda})$ with coefficient $a(\xi_{\lambda}, \xi_{\mu})$ and by Theorem 9.5 $\chi^{\Psi(\mu)}$ appears in $r'^{\beta}(\chi^{\Psi(\lambda)})$ with coefficient $-a(\xi_{\Psi(\lambda)_0}, \xi_{\Psi(\mu)_0})$. Then by Lemma 3.1 and the fact that

$$(-1)^{\frac{p^2-1}{8}}(-1)^{\frac{q^2-1}{8}} = (-1)^{\frac{(pq)^2-1}{8}},$$

the ratio of these two coefficients is

$$-(-1)^{\frac{p^2-1}{8}}\delta_{\overline{p}}(\lambda)\delta_{\overline{p}}(\mu).$$

If $\sigma(\mu) = -1$ then by Theorem 9.1 ξ_{μ}^{+} and ξ_{μ}^{-} appear in $r^{\beta}(\xi_{\lambda})$ with the same coefficient $a(\xi_{\lambda}, \xi_{\mu}^{+}) = a(\xi_{\lambda}, \xi_{\mu}^{-})$. If $\sigma(\Psi(\mu)_0) = -1$ and $\Psi(\mu)_j = \Psi(\lambda)_j$ for all $j > 0$ then by Theorem 9.5 $\chi^{\Psi(\mu)+}$ and $\chi^{\Psi(\mu)-}$ appear in $r'^{\beta}(\chi^{\Psi(\lambda)})$ with the same coefficient $-a(\xi_{\Psi(\lambda)_0}, \xi_{\Psi(\mu)_0}^{+}) = -a(\xi_{\Psi(\lambda)_0}, \xi_{\Psi(\mu)_0}^{-})$. Again the ratio of these two coefficients is

$$-(-1)^{\frac{p^2-1}{8}}\delta_{\overline{p}}(\lambda)\delta_{\overline{p}}(\mu).$$

If $\Psi(\mu)_l \neq \Psi(\lambda)_l$ for some $(1 \leq l \leq (p-1)/2)$ and $\Psi(\mu)_j = \Psi(\lambda)_j$ for all $j \neq l$ then by Theorem 9.5 $\chi^{\Psi(\mu)+}$ and $\chi^{\Psi(\mu)-}$ appear in $r'^{\beta}(\chi^{\Psi(\lambda)})$ with the same coefficient $-a(\xi_{\Psi(\lambda)_0}, \xi_{\Psi(\mu)_0}^{+}) = -a(\xi_{\Psi(\lambda)_0}, \xi_{\Psi(\mu)_0}^{-})$. The ratio of these two coefficients is

$$(-1)^{\frac{p^2-1}{8}}\delta_{\overline{p}}(\lambda)\delta_{\overline{p}}(\mu)$$

and hence we have proved $I \circ r^{\beta}(\xi_{\lambda}) = r'^{\beta} \circ I(\xi_{\lambda})$.

All the other cases are similar. We prove the most complicated case that is that the multiplicity of $\chi^{\Psi(\mu)+}$ in $I \circ r^{\beta}(\xi_{\lambda}^{+})$ is equal to its multiplicity in $r'^{\beta} \circ I(\xi_{\lambda}^{+})$ where $\sigma(\lambda) = \sigma(\mu) = -1$. In this case μ must be gotten from λ by removing a part of λ equal to pq. By Thoeroem 9.1 $\xi_{\mu}^{\pm}$ appears in $r^{\beta}(\xi_{\lambda}^{+})$ with coefficient

$$a(\xi_{\lambda}^{+}, \xi_{\mu}^{\pm}) = \frac{1}{2}(-1)^{\frac{(pq)^2-1}{8}}((-1)^{L(\overline{c}_{\mu}^{\lambda})} \pm i^{\frac{pq-1}{2}}\sqrt{pq}).$$

By Thoeroem 9.5 $\chi^{\Psi(\mu)+}$ appears in $r'^{\beta}(\chi^{\Psi(\lambda)\pm})$ with coefficient

$$a(\xi_{\Psi(\lambda)_0}^{+}, \xi_{\Psi(\mu)_0}^{+})\left(\frac{-1 \pm i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\xi_{\Psi(\lambda)_0}^{+}, \xi_{\Psi(\mu)_0}^{-})\left(\frac{-1 \mp i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)$$

if $\sigma(\Psi(\lambda)_0) = -1$ and $w - |\Psi(\lambda)_0|$ is even or

$$a(\overline{\xi}_{\Psi(\lambda)_0}^{+}, \overline{\xi}_{\Psi(\mu)_0}^{+})\left(\frac{-1 \pm i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + a(\overline{\xi}_{\Psi(\lambda)_0}^{+}, \overline{\xi}_{\Psi(\mu)_0}^{-})\left(\frac{-1 \mp i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)$$

if $\sigma(\Psi(\lambda)_0) = 1$ and $w - |\Psi(\lambda)_0|$ is odd. In either case this coefficient is

$$\begin{aligned}&\frac{1}{2}(-1)^{\frac{q^2-1}{8}}(\alpha_{\Psi(\mu)_0}^{\Psi(\lambda)_0} + i^{\frac{q-1}{2}}\sqrt{q})\left(\frac{-1 \pm i^{\frac{p-1}{2}}\sqrt{p}}{2}\right) + \frac{1}{2}(-1)^{\frac{q^2-1}{8}}(\alpha_{\Psi(\mu)_0}^{\Psi(\lambda)_0} - i^{\frac{q-1}{2}}\sqrt{q})\left(\frac{-1 \mp i^{\frac{p-1}{2}}\sqrt{p}}{2}\right)\\ &= \frac{1}{2}(-1)^{\frac{q^2-1}{8}}(-(-1)^{L(\overline{c}_{\Psi(\mu)_0}^{\Psi(\lambda)_0})} \pm i^{\frac{p+q-2}{2}}\sqrt{pq}).\end{aligned}$$

Now since

$$(-1)^{\frac{(p-1)(q-1)}{4}} i^{\frac{p+q-2}{2}} = i^{\frac{pq-1}{2}},$$

the coefficient $a(\xi_\lambda^+, \xi_\mu^\pm)$ is

$$\begin{aligned}&\frac{1}{2}(-1)^{\frac{(pq)^2-1}{8}}((-1)^{L(\overline{c}_\mu^\lambda)} \pm i^{\frac{pq-1}{2}}\sqrt{pq})\\ = -\ &\frac{1}{2}(-1)^{\frac{p^2-1}{8}}(-1)^{\frac{q^2-1}{8}}\delta_{\overline{p}}(\lambda)\delta_{\overline{p}}(\mu)\Big(-(-1)^{L(\overline{c}_{\Psi(\mu)_0}^{\Psi(\lambda)_0})} \mp \delta_{\overline{p}}(\lambda)\delta_{\overline{p}}(\mu)(-1)^{\frac{(p-1)(q-1)}{4}} i^{\frac{p+q-2}{2}}\sqrt{pq}\Big).\end{aligned}$$

Then since

$$(|\Psi(\lambda)_0| - l(\Psi(\lambda)_0)) - (|\Psi(\mu)_0| - l(\Psi(\mu)_0)) = q - 1$$

we have that the coefficient of $\chi^{\Psi(\mu)+}$ in $I \circ r^\beta(\xi_\lambda^+)$ is equal to its coefficient in $r'^\beta \circ I(\xi_\lambda^+)$. We have now proved that part (2) of Theorem 2.3 holds for G and G'. Furthermore, by Remark 2.4 part (3) also holds.

Let $x \in G$ and $x' \in G'$. We want to verify property (1) of a Broué perfect isometry. First suppose $x \in \tilde{A}_n$ and let $\Phi \in \mathbb{Z}b^\vee$ where b is a basis of $\mathbb{Z}\operatorname{Irr}(B_{x_S})^{C \cap G_{x_S}}$. By Lemma 2.1 applied to C and Theorem 4.4, $\Phi(x)$ is non-zero only if x is p-regular. So again by Lemma 2.1 applied to the set of p-regular elements we have that $\Phi\downarrow_{G_{x_S} \cap \tilde{A}_n}$ is a projective character. Then by Lemma 2.8 we have that $\hat{I}(x, x') \in |C_G(x)|\mathcal{R}$. Similarly if $x' \in \tilde{A}_{pw}$ then $\hat{I}(x, x') \in |C_{G'}(x')|\mathcal{R}$. Note that in this final case Lemma 8.4 takes the place of Theorem 4.4.

Now assume $x \notin \tilde{A}_n$ has cycle type π. Then, by Theorem 4.4, $\xi_\lambda^{(\pm)}(x) \neq 0$ only if $\pi = \lambda$. Therefore

$$\hat{I}(x, x') = \pm(\xi_\lambda^+(x)\chi^{\Psi(\lambda)+}(x') + \xi_\lambda^-(x)\chi^{\Psi(\lambda)-}(x'))$$

and so we may assume $x' \notin \tilde{A}_{pw}$. Then by Theorem 4.4 and Lemma 8.7 $\hat{I}(x, x') \in p^{l(\Psi(\lambda)_0)}\mathcal{R}$. Then by Lemma 4.2 $\hat{I}(x, x') \in |C_G(x)|\mathcal{R}$. Similarly if $x' \notin \tilde{A}_{pw}$ is of type $(\pi_0, \ldots, \pi_{(p-1)/2})$ then we may assume x is of type λ, where $\sigma(\lambda) = -1$ and

$$\hat{I}(x, x') = \pm(\xi_\lambda^+(x)\chi^{\Psi(\lambda)+}(x') + \xi_\lambda^-(x)\chi^{\Psi(\lambda)-}(x')).$$

Then by Theorem 4.4 and Lemma 8.7 either $\chi^{\Psi(\lambda)\pm}(x') \neq 0$ or $l(\pi_0) = l(\Psi(\lambda)_0)$ and $l(\pi_{p-1})$ and $\hat{I}(x, x') \in p^{l(\Psi(\lambda)_0)}\mathcal{R}$ and so, by Lemma 7.2, $\hat{I}(x, x') \in |C_{G'}(x')|\mathcal{R}$.

Now we look at property (2) of a Broué perfect isometry. If x is p-singular and x' is p-regular then by Lemma 2.6 $\hat{I}(x, x') = 0$ if $x \notin C$ so let's assume $x \in C$. Suppose x has cycle type π for some $\pi \in \mathcal{P}_n$ and then since $x \notin C$, π must have some part divisible by $2p$ and so $\pi \notin \mathcal{O}_n$. If $\pi \notin \mathcal{D}_n^-$ then by Theorem 4.4 $\xi_\lambda^{(\pm)}(x) = 0$ for all $\lambda \in \mathcal{P}_n$ and hence $\hat{I}(x, x') = 0$ so let's assume $\pi \in \mathcal{D}_n^-$. Again by Theorem 4.4 we have that $\xi_\lambda^{(\pm)}(x) \neq 0$ if and only if $\lambda = \pi$. Therefore $|\Psi(\lambda)_0| > 0$ and so by 8.5 $\chi^{\Psi(\lambda)+}(x') = \chi^{\Psi(\lambda)-}(x')$ and hence $\hat{I}(x, x') = 0$.

Next suppose x is p-regular and x' is p-singular. By Lemma 2.6 $\hat{I}(x, x') = 0$ if $x' \notin C'$ so let's assume $x' \in C'$. Let x' be of type π. Then π_0 must have some even part and so by Lemmas 8.4 and 8.7 $\chi^{\lambda\pm}(x) \neq 0$ only if $x \notin \tilde{A}_{pt}$ and $|\lambda_0| > 0$. Then $\Psi^{-1}(\lambda)$ must have some part divisible by p and so by Theorem 4.4 $\xi_{\Psi^{-1}(\lambda)}^+(x) = \xi_{\Psi^{-1}(\lambda)}^-(x)$ and hence $\hat{I}(x, x') = 0$. □

We now show that we have a Broué perfect isometry between the corresponding block of $\tilde{A}_n$ and $\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw}$.

Theorem 10.2 *The isometry*

$$\begin{aligned} I_A : \mathbb{Z}\,\mathrm{Irr}(\tilde{A}_{n,\gamma}) &\to \mathbb{Z}\,\mathrm{IrrSp}(\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw}) \\ \overline{\xi}_\lambda &\mapsto \delta_{\overline{p}}(\lambda)\delta_{\overline{p}}(\Psi(\lambda))(-1)^{w\frac{p^2-1}{8}+|\Psi(\lambda)_0|}\overline{\chi}^{\Psi(\lambda)} \qquad \text{if } \sigma(\lambda) = -1, \\ \overline{\xi}_\lambda^\eta &\mapsto \delta_{\overline{p}}(\lambda)\delta_{\overline{p}}(\Psi(\lambda))(-1)^{w\frac{p^2-1}{8}+|\Psi(\lambda)_0|}\overline{\chi}^{\Psi(\lambda)\eta} \qquad \text{if } \sigma(\lambda) = 1, \end{aligned}$$

is a Broué perfect isometry.

Proof Set $G := \tilde{A}_n$ and $G' := \tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw}$. We prove I_A is a Broué perfect isometry by studying $\hat{I}_A - \frac{1}{2}\hat{I}$ where I is taken from Theorem 10.1.

Let $x \in \tilde{A}_n$ and $x' \in \tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw}$. Note that if $\lambda \in \mathcal{D}_n^-$ then

$$\overline{\xi}_\lambda(x)\overline{\chi}^{\Psi(\lambda)}(x') - \frac{1}{2}(\xi_\lambda^+(x)\chi^{\Psi(\lambda)\pm}(x') + \xi_\lambda^-(x)\chi^{\Psi(\lambda)\mp}(x')) = 0.$$

Now if $\lambda \in \mathcal{D}_n^+$ and x is not of type λ then by Theorem 4.4

$$\overline{\xi}_\lambda^+(x)\overline{\chi}^{\Psi(\lambda)\pm}(x') + \overline{\xi}_\lambda^-(x)\overline{\chi}^{\Psi(\lambda)\mp}(x') - \frac{1}{2}\xi_\lambda(x)\chi^{\Psi(\lambda)}(x') = 0.$$

Therefore $(\hat{I}_A - \frac{1}{2}\hat{I})(x, x') = 0$ unless x is of type $\lambda \in \mathcal{D}_n^+$ and in this case

$$\begin{aligned} (\hat{I}_A - \frac{1}{2}\hat{I})(x, x') &= \pm\big(\overline{\xi}_\lambda^+(x)\overline{\chi}^{\Psi(\lambda)\pm}(x') + \overline{\xi}_\lambda^-(x)\overline{\chi}^{\Psi(\lambda)\mp}(x') - \frac{1}{2}\xi_\lambda(x)\chi^{\Psi(\lambda)}(x')\big) \\ &= \pm\frac{1}{2}(\overline{\xi}_\lambda^+(x) - \overline{\xi}_\lambda^-(x))(\overline{\chi}^{\Psi(\lambda)\pm}(x') - \overline{\chi}^{\Psi(\lambda)\mp}(x')). \end{aligned}$$

Let x' be of type π. By Lemma 8.8, $(\hat{I}_A - \frac{1}{2}\hat{I})(x, x') = 0$ unless $l(\pi_0) = l(\Psi(\lambda)_0)$ and $l(\pi_{p-1}) = 0$ and in this case, by Theorem 4.4 and Lemma 8.8, $(\hat{I}_A - \frac{1}{2}\hat{I})(x, x') \in p^{l(\Psi(\lambda)_0)}\mathcal{R}$. Then by Lemmas 4.2 and 7.2,

$$(\hat{I}_A - \frac{1}{2}\hat{I})(x, x') \in |C_G(x)|\mathcal{R} \cap |C_{G'}(x')|\mathcal{R}$$

and so $\hat{I}_A$ satisfies property (1) of a Broué perfect isometry.

Suppose x is p-regular and x' is p-singular. As above we can assume x is of type $\lambda \in \mathcal{D}_n^+$. Then $|\Psi(\lambda)_0| = 0$ and so by Lemma 8.6, $(\hat{I}_A - \frac{1}{2}\hat{I})(x, x') = 0$. Now suppose x is p-singular and x' is p-regular. Again we can assume x is of type $\lambda \in \mathcal{D}_n^+$. Then $|\Psi(\lambda)_0| > 0$ and so by Lemma 8.9 $(\hat{I}_A - \frac{1}{2}\hat{I})(x, x') = 0$. □

Before we prove Theorem 1.1 we need the following well-known lemma.

Lemma 10.3 *Let G be a finite group, H a normal subgroup and $\mathcal{R}He$ a block of $\mathcal{R}H$ such that $N_G(e) = H$. Then $f = \sum_{g \in G/H} geg^{-1}$ is a block idempotent of $\mathcal{R}G$ and $f\mathcal{R}Ge$ induces a Morita equivalence between $\mathcal{R}He$ and $\mathcal{R}Gf$.*

We are now in a position to prove Theorem 1.1 which we first restate.

Theorem. Let p be an odd prime, n a positive integer and B a p-block of $\tilde{S}_n$ or $\tilde{A}_n$ with abelian defect group. Then there exists a Broué perfect isometry between B and its Brauer correspondent.

Proof First suppose $w \geq p$. Then there is a p-subgroup of S_{pw} isomorphic to $C_p^p \rtimes C_p$ where C_p acts on C_p^p by cyclically permuting factors. This subgroup is not abelian therefore the Sylow p-subgroups of S_{pw} and hence $\tilde{S}_{pw}$ are not abelian. From now on we assume $0 < w < p$ as the result is clearly true when $w = 0$. Then by comparing orders we see that a Sylow p-subgroup Q of S_{pw} is isomorphic to C_p^w and

$$P := \{o(g) | g \in Q\} \cong C_p^w$$

is a Sylow p-subgroup of $\tilde{S}_{pw}$. Now $N_{S_{pw}}(Q) \cong N_p \wr S_w$ and so, as every element of P has odd order, $N_{\tilde{S}_{pw}}(P) \cong \tilde{N}_p^w \tilde{S}_w$. Now suppose $\gamma \vdash (n - pw)$. Then P is a defect group for $\tilde{S}_{n,\gamma}$ and

$$N_{\tilde{S}_n}(P) \cong \tilde{S}_{n-pw}(\tilde{N}_p^w \tilde{S}_w[n - pw]).$$

First suppose $\sigma(\gamma) = 1$. Then we have the following Morita equivalences

$$\begin{aligned}
&\mathcal{R}\tilde{S}_{n-pw}(\tilde{N}_p^w \tilde{S}_w[n - pw])e_{n-pw,\gamma} \\
\sim_{\text{Mor}}\ &\mathcal{R}\tilde{A}_{n-pw}(\tilde{N}_p^w \tilde{S}_w[n - pw])\overline{e}^+_{n-pw,\gamma} \\
\cong\ &\mathcal{R}\tilde{A}_{n-pw}\overline{e}^+_{n-pw,\gamma} \otimes_{\mathcal{R}} \mathcal{R}\tilde{N}_p^w \tilde{S}_w\left(\frac{1-z}{2}\right) \\
\sim_{\text{Mor}}\ &\mathcal{R}\tilde{N}_p^w \tilde{S}_w\left(\frac{1-z}{2}\right),
\end{aligned}$$

where the first equivalence is given by Lemma 10.3. Similarly,

$$\begin{aligned}
&\mathcal{R}((\tilde{S}_{n-pw}(\tilde{N}_p^w \tilde{S}_w[n - pw])) \cap \tilde{A}_n)(\overline{e}^+_{n-pw,\gamma} + \overline{e}^-_{n-pw,\gamma}) \\
\sim_{\text{Mor}}\ &\mathcal{R}\tilde{A}_{n-pw}((\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw})[n - pw])\overline{e}^+_{n-pw,\gamma} \\
\cong\ &\mathcal{R}\tilde{A}_{n-pw}\overline{e}^+_{n-pw,\gamma} \otimes_{\mathcal{R}} \mathcal{R}(\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw})\left(\frac{1-z}{2}\right) \\
\sim_{\text{Mor}}\ &\mathcal{R}(\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw})\left(\frac{1-z}{2}\right).
\end{aligned}$$

When $\sigma(\gamma) = -1$ we also have the Morita equivalences

$$\begin{aligned}
&\mathcal{R}\tilde{S}_{n-pw}(\tilde{N}_p^w \tilde{S}_w[n - pw])(e^+_{n-pw,\gamma} + e^-_{n-pw,\gamma}) \\
\sim_{\text{Mor}}\ &\mathcal{R}\tilde{S}_{n-pw}((\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw})[n - pw])e^+_{n-pw,\gamma} \\
\cong\ &\mathcal{R}\tilde{S}_{n-pw}e^+_{n-pw,\gamma} \otimes_{\mathcal{R}} \mathcal{R}(\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw})\left(\frac{1-z}{2}\right) \\
\sim_{\text{Mor}}\ &\mathcal{R}(\tilde{N}_p^w \tilde{S}_w \cap \tilde{A}_{pw})\left(\frac{1-z}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{R}((\tilde{S}_{n-pw}(\tilde{N}_p^w\tilde{S}_w[n-pw]))\cap\tilde{A}_n)\overline{e}_{n-pw,\gamma}\\
&\cong\mathcal{R}\tilde{A}_{n-pw}(\tilde{N}_p^w\tilde{S}_w[n-pw])\overline{e}_{n-pw,\gamma}\\
&\cong\mathcal{R}\tilde{A}_{n-pw}\overline{e}_{n-pw,\gamma}\otimes_{\mathcal{R}}\mathcal{R}\tilde{N}_p^w\tilde{S}_w\left(\frac{1-z}{2}\right)\\
\sim_{\mathrm{Mor}}\ &\mathcal{R}\tilde{N}_p^w\tilde{S}_w\left(\frac{1-z}{2}\right),
\end{aligned}$$

where the first isomorphism is given by

$$\begin{aligned}
\mathcal{R}\tilde{A}_{n-pw}(\tilde{N}_p^w\tilde{S}_w[n-pw])\overline{e}\ &\to\ \mathcal{R}((\tilde{S}_{n-pw}(\tilde{N}_p^w\tilde{S}_w[n-pw]))\cap\tilde{A}_n)\overline{e}\\
x\overline{e}\ &\mapsto\ \begin{cases} x\overline{e} & \text{if } x\in\tilde{A}_{n-pw}((\tilde{N}_p^w\tilde{S}_w\cap\tilde{A}_{pw})[n-pw]),\\ ix(e^+-e^-) & \text{if } x\in(\tilde{N}_p^w\tilde{S}_w\backslash\tilde{A}_{pw})[n-pw],\end{cases}
\end{aligned}$$

where $\overline{e}:=\overline{e}_{n-pw,\gamma}$ and $e^{\pm}:=e^+-e^-$.
Any Morita equivalence gives rise to a Broué perfect isometry with all signs positive (see [1, 1.2]). The result now follows from [2, Theorem 4.21] and Theorems 10.1 and 10.2. □

Acknowledgments The author thanks the referee for their comments and gratefully, acknowledges financial support by ERC Advanced Grant 291512.

References

1. Broué, M.: Isométries parfaites, types de blocs, catégories déérivées. Astérisque **181–182**, 61–92 (1990)
2. Brunat, O., Gramain, J.B.: Perfect isometries and Murnaghan-Nakayama rules. preprint: arXiv:1305.7449
3. Cabanes, M.: Local structure of the p-blocks of $\tilde{S}_n$. Math. Z. **198**(4), 519–543 (1988)
4. Chuang, J., Kessar, R.: Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture. Bull. London Math. Soc. **34**(2), 174–185 (2002)
5. Chuang, J., Rouquier, R.: Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. Ann. of Math., 245–298
6. Humphreys, J.F.: Blocks of projective representations of the symmetric groups. J. London Math. Soc. **33**(3), 441–452 (1986)
7. Isaacs, I.M.: Character Theory of Finite Groups. Academic Press, New York (1976)
8. James, G., Kerber, A.: The representation theory of the symmetric group, vol. 16. Addison-Wesley (1981)
9. Kessar, R.: Blocks and source algebras for the double covers of the symmetric and alternating groups. J. Algebra **186**(3), 872–933 (1996)
10. Michler, G.O., Olsson, J.B.: The Alperin-McKay conjecture holds in the covering groups of symmetric and alternating groups, $p \neq 2$. J. Reine und Angew. Math. **405**, 78–111 (1990)
11. Morris, A.O.: The spin representation of the symmetric group. Proc. London Math. Soc. **12**(3), 55–76 (1962)
12. Morris, A.O.: The spin representation of the symmetric group. Canad. J. Math. **17**, 543–549 (1962)
13. Morris, A.O., Olsson, J.B.: On p-quotients for spin characters. J. Algebra **119**, 51–82 (1988)
14. Olsson, J.B.: Combinatorics and representations of finite groups. Heft, 20 (1993). Vorlesungen aus dem Fachbereich Mathematik der Universitat GH Essen

15. Pfeiffer, G.: Character tables of symmetric group in GAP. Bayreuther Math. Schr. **47**, 165–222 (1994)
16. Schur, I.: Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. J. Reine und Angew. Math. **139**, 155–250 (1911)
17. Stembridge, J.R.: Shifted tableaux and the projective representations of the symmetric group. Adv. Math. **74**(1), 87–134 (1989)